Module - I: Set Theory and Matrices

Notes

Course Contents:

- Sets
- Types of sets
- Basic operations on sets
- Venn diagram
- Cartesian products on two sets
- Distributive law
- DE Morgan's laws
- Matrix
- Submatrix
- Types of Matrices
- Addition, subtraction, multiplication of matrices
- Rank of matrix

Key Learning Objectives:

At the end of this block, you will be able to:

1. Define Sets
2. Identify types of Sets
3. Describe types of Sets operations
4. Describe Venn diagram
5. Define DE Morgan's laws
6. Calculate the Cartesian product of two sets
7. Compare Matrix and Submatrix
8. Classify Matrices
9. Describe algebra of Matrices
10. Define rank of Matrix

Structure:

UNIT 1.1: Sets

1.1.1 Introduction

1.1.2 Definitions of Sets

1.1.3 Method of representation of Sets

1.1.3.1 Tabular method or Roster method

1.1.3.2 Set builder method

Notes

Notes

Notes

Unit - 1.1: Definition of Sets

Unit Outcome:

At the end of this unit, you will learn to:

- Define Sets
- Demonstrate the methods of Set representation
- Compare various types of sets
- Define subset, superset and proper subset

1.1.1 Introduction

The set theory was developed by a German Mathematician Georg Cantor (1845-1918). Nowadays, set theory is used in almost all branches of mathematics. We also use sets to define relations and functions. The knowledge of sets is required in the study of geometry, sequence, probability, etc. In this unit, we will discuss some basic definitions related to sets.

1.1.2 Definition of Sets

"A well-defined collection of objects is known as a set".

Well-defined means in a given set, it must be possible to decide whether the objects belong to the set, and by distinct, it implies that the object should not be repeated. Each object of a set is called a member or element of that set. A set is represented by { }.

Generally, sets are denoted by capital letters X, Y, Z, etc. and its elements are denoted by small letters x, y, z, etc.

Let X be a non-empty set. If x is an element of X, then we write, and it can be read as 'x is an element of X' or 'x belongs to X'. If x is not an element of X, then we write and read as 'x is not an element of X' or 'x does not belong to X'.

Example 1.1.1 Suppose we have a set X that is defined in this way

X = Set of all days in a week.

In this set, Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday are members of the set.

1.1.3 Method of representation of Sets

Sets can be represented by the following two methods:

1. Tabular method or Roster method
2. Set builder method

1.1.3.1 Tabular method or Roster method

In this method, elements are listed and put within a brace {} and separated by commas.

Example 1.1.2 Suppose we have a set X that is defined in this way

$X = \{x : x$ is an even number and x < 15)

X = {2, 4, 6, 8, 10, 12, 14}

1.1.3.2 Set builder method

In this method, instead of listing all elements of a set, we list the property or properties satisfied by the elements of a set and write it as

$X = \{x : P(x)\}$

It is read as "X is the set of all elements x such that x has the property P(x)." The symbol ':' stands for such that.

Example 1.1.3 Suppose we have a set X that is defined in this way

X = Set of all even number less than 15.

$X = \{x: x = 2n, n \in \mathbb{N} \text{ and } 1 \leq n \leq 7\}$

Or

X = {x: x is an even number less than 15}

This method is also known as Rule method.

1.1.4 Types of Sets:

(i) Empty (Void/Null) set: A set which has no element, is called an empty set. It is denoted by ϕ or {}.

Example 1.1.4 Let X = Set of all even prime numbers greater than 3.

Example 1.1.5 Let Y = Set of all prime numbers less than 2.

(ii) Singleton set: A set which has only one elements or members, is known as singleton set.

Example 1.1.6 Let X = {x: x is an even prime number} and Y = {a}

(iii) Finite set: A set which has a finite number of element or member, is known as a finite set.

Example 1.1.7 Let X = {x: x is an even number less than 9} and

Y = {1, 3, 5, 7, 11, 13, 15}

(iv) Infinite set: A set, which has an infinite number of elements or members, is known as an infinite set.

Example 1.1.8 Let X = {x: x is a natural number} and

Y = {2, 4, 6, 8, 10, 12, 14……………………}

Notes

(v) Equivalent sets: If two finite sets X and Y have the same number of elements, then the sets are known as an equivalent set.

Example 1.1.9, Let X = {2, 4, 6, 8} and Y = {1, 3, 5, 7}

(vi) Equal sets: If X and Y are two non-empty sets and each element of X is an element of set Y, and each element of set Y is an element of set X, then sets X, and Y are called equal sets.

Example 1.1.10 Let X = {x: x = 2n} and } and Y = {2, 4, 6, 8, 10}

(vii) Universal Sets: If there are some sets under consideration, then there happens to be a set which is a superset of each one of the given sets. Such a set is known as the universal set, and it is denoted by U.

Example 1.1.11 Suppose we have three sets X = {a, b}, Y = {c,d,e} Z = {f, g, h, i, j}.

$\therefore$ U = {a, b, c, d, e, f, g, h, i, j} is a universal set for all given sets.

(viii) Power sets: If X be a non-empty set, then the collection of all possible subsets of set X is known as power set. It is denoted by P(X).

The total number of elements in a power set of X containing n elements is .

Example 1.1.12

Let X = {a, b, c}

$\therefore$ P (X) = {ϕ}, {a}, {b}, {c}, {a, b}, {b, c}, {c, a}, {a, b, c}}

1.1.5 Subset and Superset

Let X and Y be two non-empty sets. If each element of set X is an element of set Y, then set X is known as a subset of set Y. If set X is a subset of set Y, then set Y is called the superset of X.

Also, if X is a subset of Y, then it is denoted as X $\subseteq$ Y and read as 'X is a subset of Y.

If $x \in X \Rightarrow x \in Y$,

then X $\subseteq$ Y

If $x \in X \Rightarrow x \notin Y$,

Then X will not be a subset of Y.

Example:1.1.13 If $X = \{a, b\}$ and $Y = \{a, b, c, d\}$

Here, each element of X is an element of Y. Thus $X \subseteq Y$, i.e. X is a subset of Y and Y is a superset of X.

1.1.6 Proper subset

If each element of X is in set Y, but set Y has at least one element which is not in X, then set X is known as a proper subset of set Y. If X is a proper subset of Y, then it is written as $X \subseteq Y$ and read as X is a proper subset of Y.

Example 1.1.14

If $N = \{1, 2, 3, 4, ...\}$

and $W = \{0, 1, 2, 3, 4, ...\}$

then $N \subseteq W$

Summary:

- A set is a well-defined collection of distinguished objects.
- Set can be represented in two ways (i) Tabular form or Roster form (ii) Set builder method.
- In the tabular form, the elements of a set are actually written down, separated by commas and enclosed within braces.
- In the set builder method, a set is described by a characterising property of its element.
- A set that does not contain a single element or member is called a null or empty set.
- A set which has only one element or member, is known as singleton set.
- A set, which has a finite number of element or member, is known as a finite set. Otherwise, it is called a non-finite set.
- Two sets X and Y are said to be equal if every element of set X is in set Y and every element of set Y is in set X.
- The collection of all subsets of a set X is called the Power set of X.
- Two sets X and Y are said to be equivalent, if the number of elements in both sets is equal.
- All the sets under consideration are likely to be subsets of a sets is called the universal set.
- A set X is called a subset of a set Y. If every element of a set X is also an element of Y and also Y is called Superset of X.
- The Powerset of a set X is the collection of all subsets of X.

Activity:

1. List ten states of India that are large in their area.
2. Now write these states in Set builder form and Tabular form.
3. Now identify what kind of set it is?
4. If set $X = \{a, b, c\}$, then find the subset of set X.
5. If set $X = \{a, b, c\}$, then find the number of element in P(X).
6. If set $X = \{a, b, c\}$, then find the number of element in P[P(X)].

Notes

Unit - 1.2: Venn Diagram

Recall Session:

In the previous unit, you studied about:

1. The definition Sets
2. The methods of Set representation
3. The various types of sets
4. Definition of subset, superset and proper subset

Unit Outcome:

At the end of this unit, you will learn to

1. Construct Venn diagram

1.2.1 Introduction

In the previous unit, we learned what subsets, supersets and proper subsets are. In this chapter, we will learn about what is a Venn diagram. With the help of the Venn diagram, we can easily solve some questions related to our everyday life.

1.2.2. Venn diagram

Most relationships between sets can be represented by diagrams called Venn diagrams. The Venn diagram is named after the English logician John Venn.

In these diagrams, rectangles and closed curves are usually circles. A universal set is usually represented by a rectangle and its subset by a circle.

In a Venn diagram, the elements of a set are written in their particular set. As shown in Fig.1.1.1.

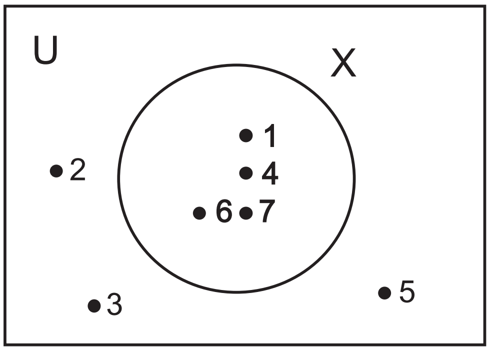


Fig.1.1.1 Venn diagram

In the fig. 1.1.1, $U = \{1, 2, 3, 4, 5, 6, 7\}$ is a Universal set and $X = \{1, 4, 6, 7\}$ is a subset.

Summary:

- Venn diagrams are diagrams that show all the possible logical relationships between finite collections of sets.

Activity:

1. Make a set of all those properties of three less than 50 that are completely divisible by 4.
2. Now draw the Venn diagram of this obtained set.

Notes

Unit - 1.3: Basic Operations on Sets

Recall Session:

In the previous unit, you studied about:

- Construction of Venn diagram
- Representation of set with the help of Venn diagram

Unit Outcomes:

At the end of this unit, you will learn to

1. Define union of sets
2. Define intersection of sets
3. Define difference of sets
4. Define disjoint sets and complement of a set
5. Define Cartesian product of two sets

1.3.1 Introduction

In the previous unit, we learned what Venn diagrams are and how to show a set with the help of a Venn diagram. As we know that if we apply an operation on a number such as a sum, difference, multiplication, and division, we get a new number. Similarly, if we apply operations on a set such as union, intersection and complement, etc., we get a new set. In this unit, we will learn how many operations are in set theory and what their properties are and how they are applied to different types of sets.

1.3.2. Types of operations on Sets.

There are mainly four types of operations in set theory which are as follows:

1. Union of sets
2. Intersection of sets
3. Difference of sets
4. Complement of sets

1.3.2.1 Union of Sets:

Suppose X and Y are any two sets. The union of X and Y is the set containing all the components of X with all the elements of Y, and the common elements are taken only once. The symbol we use to denote union. Symbolically, we write it and read it X Union Y.

Notes

The union set of the two sets X and Y is the set that contains all the elements that are either in X or in Y. Symbolically, we can write.

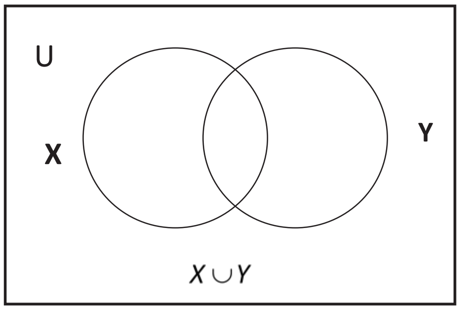


Fig.1.1.2 Union of two sets.

In the fig. 1.1.2, the total area covered by two circles X and Y are represented by $X \cup Y$.

Example 1.3.1 If $X = \{1,3,5,7\}$ and $Y = \{1,2,4,6,8\}$, then $X \cup Y = \{1,2,3,4,5,6,7,8\}$. Here we will write the number 1 only once.

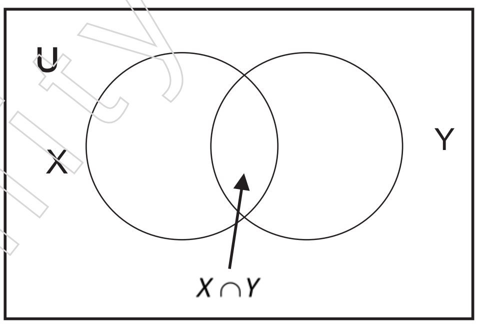
Some characteristics of operator '$\cup$'

1. $X \cup Y = Y \cup X$	(Commutative law)
2. $(X \cup Y) \cup Z = X \cup (Y \cup Z)$	(Associative law)
3. $X \cup \phi = X$	(Identity law)
4. $X \cup X = X$	(Idempotent law)
5. $U \cup X = U$	(Law of union)

1.3.2.2 Intersection of Sets:

The intersection of a set X and Y is the set of all elements that are common in both X and Y. The symbol is used to denote intersection. The set of X and Y is the common set of all the elements in both X and Y. Symbolically we can write this is as $X \cap Y$.

The intersection of the two sets X and Y is the Set of all the elements that are in both X and Y. Symbolically, we can write it as $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$.



Here, the common area of two circles is the intersection of set X and set Y.

Example 1.3.2 If $X = \{1,3,5,7\}$ and $Y = \{1,2,3,4,6,8\}$ then $X \cap Y = \{1,3\}$.

Some characteristics of operator '$\cap$'

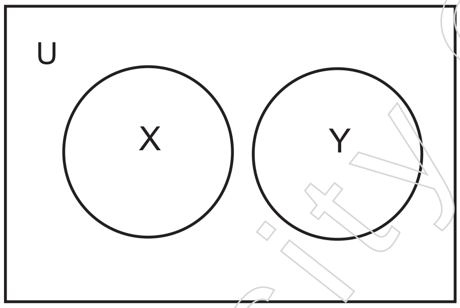
1. $X \cap Y = Y \cap X$	(Commutative law)
2. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$	(Associative law)

Notes

3. $X \cap \phi = \phi$ (Identity law)
4. $X \cap X = X$ (Idempotent law)
5. $U \cap X = X$ (Law of union)
6. $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ (Distributive law)

1.3.2.3 Disjoint Sets: Two sets X and Y are known as disjoint sets, if $X \cap Y = \phi$, i.e., if X and Y have no common element.

The Venn diagram of disjoint sets as shown in the figure:



Example 1.3.3 If $X = \{2,4,6\}$ and $Y = \{1,3,5\}$,

then $X \cap Y = \phi$. So, X and Y are disjoint sets.

1.3.2.4 Difference of two Sets:

If X and Y two non-empty sets, then difference X and Y is a set of all those elements which are in X and not in Y. It is denoted as $X - Y$. If the difference of two sets is $Y - X$, then it is a set of those elements which are in Y but not in X. Symbolically it can be written as

Hence, $X - Y = \{x : x \in X \text{ and } x \notin Y\}$.

and $Y - X = \{x : x \in Y \text{ and } x \notin X\}$.

The Venn diagram of $X - Y$ and $Y - X$ are as shown in the figure and shaded region represents $X - Y$ and $Y - X$ in figure 1.1.5 and figure 1.1.6 respectively.

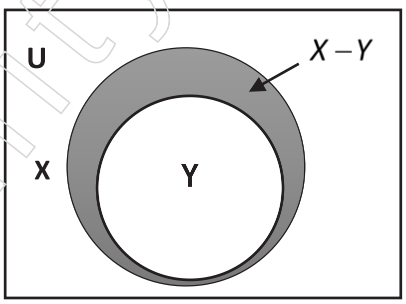


Fig. 1.1.5 Difference of two sets $X - Y$

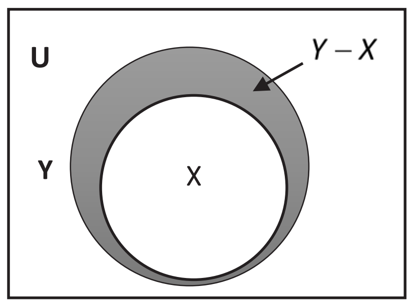


Fig. 1.1.6 Difference of two sets $Y - X$

Example 1.3.4 If X = {1,2,3,4,5,6} and Y = {1,3,5},

then $X - Y = \{2,4,6\}$.

Notes

Example 1.3.5 If Y = {*a*,*b*,*c*,*d*,*e*,*f*} and X = {*a*,*e*},

then $Y - X = \{b,c,d,f\}$

1.3.2.5 Complement of a Set:

Suppose U is a universal set and X is a subset of U, then the complementary set of X is the set of complements of U that are not components of X. Symbolically, we represent the complement of X relative to U with the symbol X' or X^C.

The Venn diagram of complement of a set X is as shown in the figure and shaded portion represents X'.

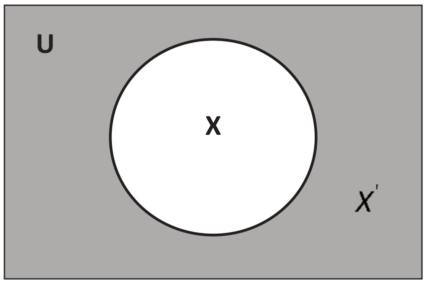


Fig.1.1.7 Complement of set X

Example 1.3.6 If $U = \{1,2,3,4,5,6,7,8\}$ and $X = \{2,4,6,8\}$, then $X' = \{1,3,5,7\}$.

Some characteristics of complementary Set

1. $X \cup X' = U$
2. $X \cap X' = \phi$
3. $(X')' = X$
4. $\phi' = U$
5. $U' = \phi$

1.3.3 Cartesian product of two sets

Let A and B be two non-empty sets. The cartesian product of A and B is denoted by and is defined as the set of all ordered pairs (a, b), where a $\in$ A and b $\in$ B.

Symbolically, $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$.

Example 1.3.7 Let $A = \{1,2,3\}$ and $B = \{x,y\}$

$\therefore A \times B = \{(1,x), (1,y), (2,x), (2,y), (3,x), (3,y)\}$

$\therefore B \times A = \{(x,1), (x,2), (x,3), (y,1), (y,2), (y,3)\}$

Summary:

- The union of two sets X and Y is the set of all those components which is either in X or in Y.
- The intersection of two sets X and Y is the set of all the elements which are common in both X or Y.
- If the intersection of two sets is, it is called disjoint sets.

Notes

- The difference of two sets X and Y is the set of all the elements that are in X nut not in Y.
- The difference of two sets Universal Set U and any set X is called the complement set of X.
- Cartesian product of two sets

  $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$.

Activity:

1. If X is the set of all months in a year and Y is the set of all month in a year which name started with J, then find the followings:
2. Union of set X and set Y
3. Intersection of set X and set Y
4. $X - Y$
5. $Y - X$
6. Complement of set X

Unit - 1.4: DE Morgan's law

Recall Session:

In the previous unit, you studied about:

- The union of sets.
- The intersection of sets
- The disjoint sets
- The difference between sets
- The complement of a set
- The cartesian product of two sets

Unit Outcome:

At the end of this unit, you will learn to

1. Define DE Morgan's law
2. Define Distributive law

1.4.1 Introduction

In the previous unit, we learned about the various types of operations of a set. In this unit, we will learn about what is DE-Morgan's law.

1.4.2 DE Morgan's law

According to this rule, "The complement of the union of the two sets is the intersection of their complementary sets." And "The complement of intersection of the two sets is the union of their complementary sets." Symbolically, we represent it as

1. $(X \cup Y)' = X' \cap Y'$
2. $(X \cap Y)' = X' \cup Y'$

Example. 1.4.1 Let $U = \{1,2,3,4,5,6\}$, $A = \{2,3\}$ and $B = \{3,4,5\}$. Show that $(A \cup B)' = A' \cap B'$.

Answer:

Given U = {1,2,3,4,5,6}

$A = \{2,3\}$

$B = \{3,4,5\}$

$A \cup B = \{2,3,4,5\}$

$(A \cup B)' = \{1,6\}$

Also, $A' = \{1,4,5,6\}$

Notes

$B' = \{1,2,6\}$

$A' \cap B' = \{1,6\}$

Hence, $(A \cup B)' = A' \cap B'$

Example 1.4.2 If A and B are two sets, then find the value of $A \cap (A \cup B)'$.

Answer:

$A \cap (A \cup B)' = A \cap (A' \cap B')$ [by De-Morgan's Law]

$= (A \cap A') \cap B'$ [by associative Law]

$= \phi \cap B'$ $[\therefore A \cap A' = \phi]$

$= \phi$

1.4.3 Distributive laws

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example 1.4.4. If A and B are non-empty sets, then find the value of.

Solution:

$(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B')$ $[\therefore A - B = A \cap B]$

$= A \cap (B \cap B')$ [by distributive Law]

$= A \cap U$ $(\therefore B \cap B' = U)$

$= A$

Example 1.4.5 For any two sets A and B, prove the following:

$A \cap (A' \cup B) = A \cap B$

Solution:

From L.H.S.

$= A \cap (A' \cup B)$

$= (A \cap A') \cup (A \cap B)$

$= \phi \cap (A \cap B)$ $[\therefore A \cap A' = \phi]$

$= A \cap B$

Hence Proved.

Example 1.4.6 If $A = \{1,2,3,6\}$, $B = \{2,4,6,8\}$ and $C = \{1,3,5,7\}$, then find the value of

a. $(A \cup B) \cap (A \cup C)$
b. $(A \cap B) \cup (A \cap C)$

Solution:

(a) $(A \cup B) \cap (A \cup C) = (A \cap B) \cup (A \cap C)$ [by distributive law]

Notes

$= \{1,2,3,6\} \cap [\{2,4,6,8\} \cap \{1,3,5,7\}]$

$= \{1,2,3,6\} \cup \phi$

$= \{1,2,3,6\}$

$= A$

(b) $(A \cup B) \cap (A \cup C) = A \cap (B \cup C)$ [by distributive law]

$= \{1,2,3,6\} \cap [\{2,4,6,8\} \cup \{1,3,5,7\}]$

$= \{1,2,3,6\} \cup \{1,2, 3, 4, 5, 6, 7, 8\}$

$= \{1,2,3,6\}$

$= A$

Summary:

- **De-Morgan's law**
  - $(X \cup Y)' = X' \cup Y'$
  - $(X \cap Y)' = X' \cup Y'$
- **Distributive laws**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Activity:

a. If X and Y are two sets, then find the value of.

Notes

Unit - 1.5: Matrix

Recall Session:

In the previous unit, you studied about:

- DE Morgan's law

Unit Outcome:

At the end of this unit, you will learn to

1. Define Matrix
2. Describe various types of Matrices
3. Describe Algebra of Matrices
4. Rank of Matrix

1.5.1 Introduction

In the previous unit, we learned about De Morgan's law and in the earlier units we have learned the union, intersection and difference of the two sets.

Knowledge of matrices is required in various branches of mathematics. Matrix is one of the most powerful tools of mathematics. Compared to other straightforward methods, this mathematical tool makes our work much easier. The concept of the matrix evolved as an attempt to solve the system of linear equations in a short and simple form. Matrix notation or representation and operations are used to create electronic spreadsheet programs that are used in various fields of commerce and science like budgeting, sales projection, cost estimation, analysis of the result of an experiment, etc. In this unit, we will discuss the matrix, equal matrix and its various types.

1.5.2 Matrix

An arrangement of mn numbers or functions in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of the type m by n (or m×n).

Such an array is enclosed by the bracket [ ].

Each number constituting the matrix is called an element of the matrix.

The location of each element in the matrix is fixed. Therefore, the elements of the matrix are represented by letters which have two subscripts. The first subscript row and the second subscript column reveal which row and which column the element is in. Thus, the element in the i^{th} row and j^{th} column is written with a_{ij}. Therefore, the matrix in the m row and n column is often written as follows

$$A_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

This is called a matrix of $m \times n$.

The first letter of the $m \times n$ represents the number of rows in the matrix A, and the second letter, the number of its columns.

Hence, we have

$\begin{bmatrix} 2 & 8 & 3 \\ 5 & 7 & 4 \end{bmatrix}$ is a 2 × 3 matrix, and $\begin{bmatrix} 2 & 5 \\ 8 & 7 \\ 3 & 4 \end{bmatrix}$ a 3 × 2 matrix.

1.5.3 Types of Matrices

1.5.3.1 Row and Column matrix- A matrix having only one row is called a row matrix while the one having only one column is known as a column matrix.

Example 1.5.1 [2 6 3] is a (1 × 3) row matrix while $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ is a (3 × 1) column matrix.

1.5.3.2 Null Matrix- An $m \times n$ matrix each of whose elements is 0, is called a null matrix of the type.

Example 1.5.2 $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are null matrices of the type 2 × 3, 3 × 3 and 2 × 2 respectively.

1.5.3.3 Square Matrix- An $m \times n$ matrix for which $m = n$, i.e., the number of rows equals the number of columns is called a square matrix of order n or an n-rowed square matrix.

The elements a_{ij} of a square matrix, $A = [a_{ij}]_{n\times n}$ for which i = j, are called the diagonal elements of the matrix and the line along which they lie is called or simply the diagonal of the matrix.

Example 1.5.3 $\begin{bmatrix} 1 & 9 & 25 \\ 4 & 12 & 30 \\ 8 & 16 & 32 \end{bmatrix}$ is a square matrix of order 3.

1.5.3.4 Diagonal Matrix- A square matrix in which each one of the non-diagonal matrices is 0, is called a diagonal Matrix.

Notes

Example 1.5.4 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix of order 3 × 3.

1.5.3.5 Rectangular Matrix- A m×n matrix in which the number of rows and columns is not same, is called a rectangular matrix.

Example 1.5.5 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ is a rectangular matrix of order 2×3 .

1.5.3.6 Scalar Matrix- A diagonal matrix in which all the diagonal elements are equal is called a scalar matrix.

Example 1.5.6 $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a scalar matrix of order 3×3.

1.5.3.7 Symmetric matrix- A square matrix is called a symmetric matrix if for every value of i and j $a_{ij} = a_{ji}$.

Example 1.5.7 $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ is a symmetric matrix of order 3×3.

1.5.3.8 Skew – Symmetric Matrix – A square matrix is said to be skew – symmetric if $A' = -A$ i.e., if $A = [a_{ij}]_{n,}$ then A is called skew symmetric matrix.

Example. 1.1.8 $\begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}$ is a skew – symmetric matrix.

1.5.3.9 Unit or identity matrix- Scalar matrix each of whose diagonal element is unity, is called a unit matrix or an identity matrix. It is represented by 'I'.

Example 1.5.8 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an Identity matrix of order 2×2 and 3×3, respectively.

1.5.3.10 Upper Triangular matrix- A square matrix in which each element below the principal diagonal is 0, is called an upper triangular matrix.

Example 1.5.9 $\begin{bmatrix} 4 & 2 & 6 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ is called an upper triangular matrix of order 3×3.

Notes

1.5.3.11 Lower triangular matrix- A square matrix in which each element above the principal diagonal is 0, is called a lower triangular matrix.

Example 1.5.10 $\begin{bmatrix} 4 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 4 & 6 \end{bmatrix}$ is a lower triangular matrix of order 3×3.

1.5.3.12 Singular Matrix

A square matrix is called singular if $|A| = 0$.

Example 1.5.11 Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$

$$|A| = 8 - 8$$
$$= 0$$

Hence A is a singular matrix.

1.5.3.13 Non-singular Matrix

A square matrix is called a non-singular if $|A| \neq 0$.

Example 1.5.12 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$|A| = 4 - 6$$
$$= -2$$
$$\neq 0$$

Hence, A is a non-singular matrix.

1.5.3.14 Equal Matrices- Two Matrices A and B are said to be, written as $A = B$, if they are of the same type and their corresponding elements are equal.

Example 1.5.13 $\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ are not equal but $\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$ and $\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$ are equal matrix.

Summary:

- An arrangement of $m \times n$ numbers or functions in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of the type m by n (or $m \times n$).
- A matrix having only one row is called a row matrix while the one having only one column is known as a column matrix.
- A $m \times n$ matrix each of whose elements is 0, is called a null matrix of the type $m \times n$.

Notes

- A $m \times n$ matrix for which $m = n$, i.e., the number of rows equals the number of columns is called a square matrix of order n or an n-rowed square matrix.
- A square matrix in which each one of the non-diagonal matrices is 0, is called a diagonal Matrix.
- A matrix in which the number of rows and columns is uneven, is called a rectangular matrix.
- A diagonal matrix in which all the diagonal elements are equal is called a scaler matrix.
- A square matrix is called a symmetric matrix if for every value of i and j $a_{ij} = a_{ji}$.
- Scalar matrix each of whose diagonal element is unity, is called a unit matrix or an identity matrix. It is represented by 'I'.
- A square matrix in which each element below the principal diagonal is 0, is called an upper triangular matrix.
- A square matrix in which each element above the principal diagonal is 0, is called a lower triangular matrix.
- A square matrix is called a singular if $|A| = 0$.
- A square matrix is called a non-singular if $|A| \neq 0$.
- Two matrices A and B are said to be, written as $A = B$, if they are of the same type and their corresponding elements are equal.

Unit - 1.6: Algebra of a Matrices and Rank of Matrix

Recall Session:

In the previous unit, you studied about:

- Define matrix
- Describe the various types of matrices
- Define Equal matrices

Unit Outcome:

At the end of this unit, you will learn to

1. Describe algebra of matrices
2. Describe properties of algebra of matrices
3. Define transpose of a matrix
4. Describe properties of transpose of a matrix
5. Define Rank of Matrix

1.6.1 Introduction

In the previous unit, we studied about matrix, various types of matrices, and equal matrices. In this unit, we will learn about algebra of matrices, properties of algebra of matrices, the transpose of a matrix and the properties of the transpose of a matrix also.

1.6.2 Algebra of a Matrices

1.6.2.1 Addition of Matrices

If two matrices A and B are equal in number of rows and the number of columns is also equal, then *A* + *B* is the matrix whose each element is equal to the sum of the corresponding elements of matrices *A* and *B*. Thus if

$$A=\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix},\ B=\begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}$$

then, $$A+B=\begin{bmatrix} a_1+c_1 & a_2+c_2 \\ b_1+d_1 & b_2+d_2 \end{bmatrix}$$

1.6.2.2 Difference of two matrices

If two matrices A and B are equal in number of rows and the number of columns is also equal, then is the matrix whose each element is equal to the difference of the corresponding components of A and B. Thus if

Notes

$$A=\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, B=\begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}$$

$$\text{then, } A-B=\begin{bmatrix} a_1-c_1 & a_2-c_2 \\ b_1-d_1 & b_2-d_2 \end{bmatrix}$$

1.6.2.3 Multiplication of two Matrices

If A and B are two such matrices that the number of columns of A is equal to the number of rows of B, then the multiplication of A and B matrix will be AB.

$$\text{Thus if } A=\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, B=\begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}$$

$$\text{then, } AB=\begin{bmatrix} a_1c_1+a_2d_1 & a_1c_2+a_2d_2 \\ b_1c_1+b_2d_1 & b_1c_2+b_2d_2 \end{bmatrix}$$

1.6.2.4 Negative Matrix

A negative matrix of $A = [a_{ij}]$ denoted by $-A$ where $-A = [-a_{ij}]$.

$$\text{Thus if } A=\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, \text{ then } -A=\begin{bmatrix} -a_1 & -a_2 \\ -b_1 & -b_2 \end{bmatrix}$$

1.6.3 Properties of Matrix Addition

Suppose A, B, C three matrices are of the same order .

i. Commutative Law: $A + B = B + A$

ii. Associative Law: $A + (B + C) = (A + B) + C$

iii. Additive Identity: $A + O = A = O + A$

where O is null matrix of order $m \times n$.

iv. Additive inverse: $A + (-A) = O$

$-A$ is called the additive inverse of A.

v. Cancellation Law: If $A + B = A + C$

then $B = C$

vi. Properties of Scaler Multiplication: $k(A + B) = kA + kB$ and $(k_1 + k_2)A = k_1A + k_2A$, where k, k_1 and k_2 are constant or scaler quantity.

1.6.4 Properties of Multiplication of Matrices

Suppose A, B, C are three matrices.

i. Associative Law: $(AB).\underline{C} = A.(BC)$

ii. Distributive Law:

a. $A(B + C) = AB + BC$

b. $(A + B)C = AC + BC$

iii. Identity Law: $IA = AI = A$

1.6.5 Transpose of a Matrix

The transpose of a matrix is a matrix formed, from the original where the rows of the original matrix are the column of the transpose matrix. The transpose of matrix A is represented by.

$$\text{Thus if } A=\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \text{ then } A'=\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}.$$

1.6.6 Properties of Transpose of Matrices

a. $(A')' = A$

b. $(kA)' = kA'$

c. $(A + B)' = A' + B'$

d. $(AB) = B'A'$

1.6.7 Rank of a matrix

It is said that the rank of matrix A is r if

a. Every minor of A of order $r + 1$ is zero.

b. There is at least one minor of A of order r which does not vanish.

Or

It is said that the rank of matrix A is r if

a. Every square sub-matrix of order $r + 1$ is singular.

b. There is at least one square sub-matrix of order r which is non- singular.

The rank of a matrix can be represented by the symbol 'ρ'.

Summary:

- If two matrices A and B are equal in the number of rows and the number of columns is also equal, then $A + B$ is the matrix whose each element is equal to the sum of the corresponding elements of matrices A and B.
- If two matrices A and B are equal in the number of rows and the number of columns is also equal, then $A - B$ is the matrix whose each element is equal to the difference of the corresponding components of A and B.
- If A and B are two such matrices that the number of columns of A is equal to the number of rows of B, then the multiplication of A and B matrix will be AB.
- The negative form of the matrix $A = -A$.
- The transpose of a matrix is a matrix formed, from the original where the rows of the original matrix are columns of the transpose matrix. The transpose of matrix A is represented by A'.

Notes

- It is said that the rank of matrix A is r if
  a. Every square sub-matrix of order $r + 1$ is singular.
  b. There is at least one square sub-matrix of order r which is non- singular.

Solved Examples:

Ex-1.1 Check which of the following are sets and why?

a. The collection of all the student in your class.
b. The collection of the world's biggest animal.
c. The collection of all vowels in the English alphabet.
d. The collection of five best Bollywood actors of India.
e. The collection of all whole number.

Solution:

a. The collection of all the student in your class is a well-defined collection because you can definitely identify those students who belong to this collection.

   So, this collection is a well-defined set.

b. The collection of the world's biggest animal is not a well-defined collection because the criteria for determining the biggest animal can vary from person to person.

   So, this collection is not a well-defined set.

c. The collection of all vowels in the English alphabet is a well-defined collection because you can definitely identify those alphabets who belongs to this collection.

   So, this collection is a well-defined set.

d. The collection of five best Bollywood actors in India is not a well-defined collection because the criteria for determining the five best Bollywood actors can vary from person to person.

   So, this collection is not a well-defined set.

e. The collection of all whole number in your class is a well-defined collection because you can definitely identify those numbers who belong to this collection.

   So, this collection is a well-defined set.

Ex-1.2 Write the following sets in set builder method:

a. $X = \{4,8,12,16,20\}$
b. $X = \{4,8,12,16,20\}$
c. $X = \{a,e,i,o,u\}$
d. $X = \{1,8,27,64,125\}$
e. $X = \{1,3,5,7,9\}$

Solution:

a. $X = \{x{:}x = 4n,\ n \in N$ and $1 \le n \le 5\}$
b. $X = \{x{:}x$ is a prime number less than 12$\}$
c. $X = \{x{:}x$ is a vowel of the English alphabets$\}$
d. $X = \{x{:}x = n^3,\ n \in N$ and $1 \le n \le 5\}$

e. $X = \{x{:}x$ is an odd number less than 10$\}$

Ex-1.3 Write the following sets in the roster method:

a. $X = \{x{:}x$ is an even number less than 10$\}$

b. $X = \{x{:}x$ is an even prime number$\}$

c. $X = \{x{:}x$ is a multiple of 5$\}$

d. $X = \{x{:}x$ is a positive integer$\}$

e. X = The set of all letters in the word AMITY

Solution:

a. $X = \{2,4,6,8\}$

b. $X = \{2\}$

c. $X = \{5,10. 15,20,......\}$

d. $X = \{1,2,3,4,5,......\}$

e. $X = \{A,M,I,T,Y\}$

Ex-1.4 Which of the following sets are finite or infinite or empty:

a. A set of real numbers.

b. The set of whole numbers which are less than 25.

c. The set of numbers multiple of 11.

d. The set of odd numbers which are divisible by 2.

e. The collection of all professors in your college.

f. The set of negative natural numbers.

g. The set of composite numbers.

Solution:

a. A set of real numbers is an infinite set because it has an infinite number of real numbers.

b. The set of whole numbers which are less than 25 is a finite set because it has a finite number of whole numbers.

c. The set of numbers multiple of 11 is an infinite set because it has an infinite number of elements.

d. The set of odd numbers which are divisible by 2 is an empty set because it has 0 elements.

e. The collection of all professors in your college is a finite set because it has a finite number of professors.

f. The set of negative natural numbers is an empty set because it has 0 elements.

g. The set of composite numbers is an infinite set because it has an infinite number of elements.

Ex-1.5 Write all the subsets of the following sets:

a. $\{b\}$

b. $\{2,3,5\}$

c. $\{a,1,2\}$

d. $\{a,b,c\}$

Solution:

a. Subsets of $\{b\} = \phi, \{b\}$

Notes

b. Subsets of {2,3,5} = ϕ, {2}, {3}, {5}, {2,3}, {3,5}, {2,5}, {2,3,5}
c. Subsets of {*a*,1,2} = ϕ, {*a*}, {1}, {2}, {*a*,1}, {1,2}, {*a*,2}, {*a*,1,2}
d. Subsets of {*a*,*b*,*c*} = ϕ, {*a*}, {*b*}, {*c*}, {*a*,*b*}, {*b*,*c*}, {*a*,*c*}, {*a*,*b*,*c*}

Ex-1.6 Write all the proper subsets of the Q 1.4:

Solution:

a. Proper Subsets of {*b*} = {ϕ, {*b*}}
b. Proper Subsets of {2,3,5} = {ϕ, {2}, {3}, {5}, {2,3}, {3,5}, {2,5}, {2,3,5}}
c. Proper Subsets of {*a*,1,2} = {ϕ, {*a*}, {1}, {2}, {*a*,1}, {1,2}, {*a*,2}, {*a*,1,2}}
d. Proper Subsets of {*a*,*b*,*c*} = {ϕ, {*a*}, {*b*}, {*c*}, {*a*,*b*}, {*b*,*c*}, {*a*,*c*}, {*a*,*b*,*c*}}

Ex-1.7 Write a universal set for the following sets:

a. X = {2,3,5}, Y = {1,4,6} and Z = {1,8,9}
b. X = {a,b}, Y = {c,d,e} and Z = {e, f}
c. X = {1, a},Y = {2,b} and Z = {3,c}
d. X = {Sunday, Monday}, Y = {Friday, Saturday} and Z = {Tuesday}
e. X = {a,i,o}, Y = {e,u} and Z = {a,e}

Solution:

a. U = {1,2,3,4,5,6,8,9}
b. U = {a,b,c,d,e, f}
c. U = {a,b,c,1,2,3}
d. U = {Friday,Saturday,Sunday,Monday,Tuesday}
e. U = {a,e,i,o,u}

Ex-1.8 If A = {2,3,5}, B = {1,4,6},C = {7,8,9} and D = {10,11,12}, find the following sets:

a. $A \cup B$
b. $A \cup C$
c. $A \cup D$
d. $B \cup C$
e. $B \cup D$
f. $A \cup B \cup C$
g. $A \cup B \cup D$
h. $B \cup C \cup D$

Solution:

a. $A \cup B$ = {1,2,3,4,5,6}
b. $A \cup C$ ={2,3,5,7,8,9}
c. $A \cup D$ ={2,3,5,10,11,12}
d. $B \cup C$ ={1,4,6,7,8,9}
e. $B \cup D$ ={1,4,6,10,11,12}
f. $A \cup B \cup C$ ={1,2,3,4,5,6,7,8,9}

g. $A \cup B \cup D = \{1,2,3,4,5,6,10,11,12\}$

h. $B \cup C \cup D = \{1,4,6,7,8,9,10,11,12\}$

Ex-1.9 If $A = \{3,6,9,12,15\}$, $B = \{4,8,12,16,20\}$, $C = \{2,4,6,8,10,12\}$ and $D = \{5,10,15,20,25\}$, find the following sets:

a. $A - B$

b. $A - C$

c. $A - D$

d. $B - C$

e. $B - D$

f. $B - A$

g. $C - A$

h. $C - B$

i. $C - D$

j. $D - A$

k. $D - B$

l. $D - C$

Solution:

a. $A - B = \{3,6,9,15\}$

b. $A - C = \{3,9,15\}$

c. $A - D = \{3,6,9,12\}$

d. $B - C = \{16,20\}$

e. $B - D = \{4,8,12,16\}$

f. $B - A = \{4,8,16,20\}$

g. $C - A = \{2,4,8,10\}$

h. $C - B = \{2,6,10\}$

i. $C - D = \{2,4,6,8,12\}$

j. $D - A = \{5,10,20,25\}$

k. $D - B = \{5,10,15,25\}$

l. $D - C = \{5,15,20,25\}$

Ex-1.10 If $A = \{1,3,5,7\}$, $B = \{3,5,7,9\}$, $C = \{3,7,9,11\}$, and $D = \{1,5,9,11\}$, find the following sets:

a. $A \cap B$

b. $A \cap C$

c. $B \cap C$

d. $A \cap B \cap C$

e. $B \cap C \cap D$

f. $A \cap D$

g. $A \cap (B \cup D)$

h. $(A \cap B) \cap (B \cup C)$

i. $(A \cup D) \cap (B \cup C)$

Notes

Solution:

a. $A \cap B = \{3,5,7\}$

b. $A \cap C = \{3,7\}$

c. $B \cap C = \{3,7,9\}$

d. $A \cap B \cap C = \{3,7\}$

e. $B \cap C \cap D = \{11\}$

f. $A \cap D = \{1,5\}$

g. $A \cap (B \cap D) = \{1,3,5,7\}$

h. $(A \cap B) \cap (B \cup C) = \{3,5,7\}$

i. $(A \cup D) \cap (B \cap C) = \{3,5,7,9,11\}$

Ex-1.11 If U = {1,2,3,4,5,6,7,8,9,10} and A = {4,5,6,7}, find the value of A^C .

Solution:

$\mathbf{A^C} = U - A$

$= \{1, 2, 3, 8, 9, 10\}$

Ex-1.12 If U = {1,2,3,4,5,6,7,8,9,10}, A = {4,5,6,7} and B = {1,2,3,9}, find the value of followings:

a. B^c

b. $(A - B)^c$

c. $(B - A)c$

d. $(A^c)^c$

e. $(A \cup B)^c$

Solution:

a. The complement of B is

$B^c = U - B$

$= \{4,5,6,7,8,10\}$

b. $A - B = \{4,5,6,7\}$

$(A - B)^c = U - (A - B)$

$= \{1,2,3,8,9,10\}$

c. $B - A = \{1,2,3,9\}$

$(B - A)^c = U - (B - A)$

$= \{4,5,6,7,8,10\}$

d. $A^c = \{1,2,3,8,9,10\}$

$(A^c)^c = \{4,5,6,7\}$

e. $A \cup B = \{1,2,3,4,5,6,7,9\}$

$(A \cup B)^c = \{8,10\}$

Ex 1.13 In a survey of 400 movie viewers, 150 were listed as liking 'Veer-Zara', 100 as liking 'Aitraaz' and 75 were listed as both liking 'Aitraaz' as well as 'Veer-Zara'. Find how many people were liking neither 'Aitraaz' nor 'Veer-Zara'.

Answer: Let U denote the set of movie viewers. Let V denote the set of people liking 'Veer-Zara' and A denote the set of people liking 'Aitraaz'. Then

n (U) = 34

n (V) = 150

n (A) = 100

n (V $\cap$ A) = 75

Now,

n (V' $\cap$ A') = n (V $\cup$ A)'

= n (U) – n(V $\cup$ A)

= n (U) – n(V) – n (A) + n (V $\cap$ A)

= 400 – 150 – 100 + 75

= 225

Ex-1.1.4 In a town of 10,000 families, it was found that 40% families buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C, 5% buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all of three newspapers, then find the number of families which buy A only.

Solution: Given: -

$$n(A) = 40\% \text{ of } 10{,}000$$
$$=4000$$
$$n(B) = 20\% \text{ of } 10{,}000$$
$$=2000$$
$$n(C) = 10\% \text{ of } 10{,}000$$
$$=1000$$
$$n(A\cap B) = 5\% \text{ of } 10{,}000$$
$$=500$$
$$n(B\cap C) = 3\% \text{ of } 10{,}000$$
$$=300$$
$$n(C\cap A) = 4\% \text{ of } 10{,}000$$
$$=400$$
$$n(A\cap B\cap C) = 2\% \text{ of } 10{,}000$$
$$=200$$

Notes

So,

$$n(A \cap B' \cap C') = n\{A \cap (B \cup C)'\} \qquad [\because \text{ by De-Morgan's law}]$$
$$= n(A) - n\{A \cap (B \cup C)\}$$
$$= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$
$$= 4000 - 500 - 400 + 200$$
$$= 3300$$

Ex- 1.15 Out of 100 students; 15 passed in English, 12 passed in Mathematics, 8 in Science, 6 in English and Mathematics, 7 in Mathematics and Science, 4 in English and Science and 4 in all the three passed. Find the number of students who only passed in Mathematics.

Solution: Let E, M and S be denoting the total number of students passed in English, passed in Mathematics and passed in Science, respectively.

$$n(U) = 100$$
$$n(E) = 15$$
$$n(M) = 12$$
$$n(S) = 8$$
$$n(E \cap M) = 6$$
$$n(M \cap S) = 7$$
$$n(E \cap S) = 4$$
$$n(E \cap M \cap S) = 4$$

So,

$$n(M \cap E' \cap S') = n\{M \cap (E \cup S)'\} \qquad [\because \text{ by De-Morgan's law}]$$
$$= n(M) - n\{M \cap (E \cup S)\}$$
$$= n(M) - n(M \cap E) - n(M \cap S) + n(M \cap E \cap S)$$
$$= 12 - 6 - 7 + 4$$
$$= 3$$

Ex- 1.16 If $U = \{a,b,c,d,e\}$, $A = \{a,b,d\}$ and $\text{B} = \{b,d,e\}$. Prove that the De-Morgan's law of intersection.

Solution:

$$U = \{a,b,c,d,e\}$$
$$A = \{a,b,d\}$$
$$B = \{b,d,e\}$$

$$A \cap B = \{a,b,d\} \cap \{b,d,e\}$$
$$A \cap B = \{b,d\}$$

Notes

From equation (1) and (2)

$$\text{So,}\quad (A\cap B)' = \{a,c,e\} \qquad ....(1)$$

$$A' = \{c,e\}$$

$$B' = \{a,c\}$$

$$\text{So,}\quad A' \cup B' = \{c,e\} \cup \{a,c\}$$

$$= \{a,c,e\} \qquad ....(2)$$

From equation (1) and (2)

$$(A\cap B)' = A' \cup B'$$

Which is a De-Morgan's law of intersection.

Hence Proved.

Ex- 1.17 If $A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 4 \\ -1 & 8 \end{bmatrix}$, then find the followings:

(a) $A+B$

(b) $A-B$

(c) $B+C$

(d) $2B-C$

(e) AB

(f) BA

Solution:

(a) Given: $A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

So, $A+B = \begin{bmatrix} 6 & 7 \\ 1 & 7 \end{bmatrix}$

(b) Given: $A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

So, $A-B = \begin{bmatrix} 4 & 1 \\ 5 & -3 \end{bmatrix}$

Notes

(c) Given: $B=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ and $C=\begin{bmatrix} 2 & 4 \\ -1 & 8 \end{bmatrix}$

So, $B+C=\begin{bmatrix} 3 & 7 \\ -3 & 13 \end{bmatrix}$

(d) Given: $B=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ and $C=\begin{bmatrix} 2 & 4 \\ -1 & 8 \end{bmatrix}$

So,

$$2B-C=\begin{bmatrix} 2 & 6 \\ -4 & 10 \end{bmatrix}-\begin{bmatrix} 2 & 4 \\ -1 & 8 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 2 \\ -3 & 2 \end{bmatrix}$$

(e) Given: $A=\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

So,

$$AB=\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$$

$$=\begin{bmatrix} 5-8 & 15+20 \\ 3-4 & 9+10 \end{bmatrix}$$

$$=\begin{bmatrix} -3 & 35 \\ -1 & 19 \end{bmatrix}$$

(f) Given: $A=\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

So,

$$BA=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 5+9 & 4+6 \\ -10+15 & -8+10 \end{bmatrix}$$

$$=\begin{bmatrix} 14 & 10 \\ -5 & 2 \end{bmatrix}$$

Notes

Ex-1.18 If $x\begin{bmatrix}4\\6\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}20\\10\end{bmatrix}$, then find the value of x and y.

Solution: Given:

$$x\begin{bmatrix}4\\6\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}20\\10\end{bmatrix}$$

$$\because \begin{bmatrix}4x-2y\\6x+2y\end{bmatrix}=\begin{bmatrix}20\\10\end{bmatrix}$$

$$\Rightarrow 4x-2y=20 \quad ....(1)$$

$$\Rightarrow 6x+2y=10 \quad ....(2)$$

On solving,

10x = 30

x = 3

Substitute the value of x into equation (1)

12 – 2y = 20

– 2y = 3

Hence, the value of x and y are 3 and -4, respectively.

Ex-1.19 Find the transpose of following matrix :

(a) $A=\begin{bmatrix}1 & 1\\2 & 4\\3 & 9\end{bmatrix}$

(b) $B=\begin{bmatrix}1 & -6\\3 & 8\end{bmatrix}$

(c) $C=\begin{bmatrix}a & d & e\\b & f & g\\c & h & i\end{bmatrix}$

(d) $D=\begin{bmatrix}4 & 12 & 20\\5 & 15 & 25\end{bmatrix}$

Notes

Solution:

(a) Given: $A=\begin{bmatrix}1 & 1\\ 2 & 4\\ 3 & 9\end{bmatrix}$

$$A'=\begin{bmatrix}1 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}$$

(b) Given: $B=\begin{bmatrix}1 & -6\\ 3 & 8\end{bmatrix}$

$$B'=\begin{bmatrix}1 & 3\\ -6 & 8\end{bmatrix}$$

(c) Given: $C=\begin{bmatrix}a & d & e\\ b & f & g\\ c & h & i\end{bmatrix}$

$$C'=\begin{bmatrix}a & b & c\\ d & f & h\\ e & g & i\end{bmatrix}$$

(d) Given: $D=\begin{bmatrix}4 & 12 & 20\\ 5 & 15 & 25\end{bmatrix}$

$$D'=\begin{bmatrix}4 & 5\\ 12 & 15\\ 20 & 25\end{bmatrix}$$

Ex-1.20 Find the rank of the following matrix :

(a) $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 0 & 2 & 2\end{bmatrix}$

(b) $B=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 3 & 5 & 7\end{bmatrix}$

Solution:

(a) Given: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

The determinant of A for order 3

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{vmatrix}$$

$$= 1(6-8) - 2(0-4) + 3(4-0)$$

$$= -2 + 8 + 12$$

$$= 18$$

$$\neq 0$$

Hence, rank of matrix A is 3.

Given: $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

Determinant of A for order 3

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix}$$

$$= 1(21-20) - 2(14-12) + 3(10-9)$$

$$= 1 - 4 + 3$$

$$= 0$$

$$\because |B| = 0$$

Hence, rank of matrix A cannot be 3.

So, we check the any minor of order 2 is the determinant of order 2, i.e.,

$$|B| = \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix}$$

$$= 21 - 20$$

$$= 1$$

$$\because |B| \neq 0$$

Hence, rank of matrix A is 2.

Notes

Further Readings:

- Class XI & XII N.C.E.R.T Mathematics Book
- Ravita Bharadwaj, "Mathematics for Managers", University Science Press, New Delhi
- P.C. Tulsian, Bharat Tulsian, "Tulsian's Business Mathematics, Logical Reasoning & Statistics", McGraw Hill Education Private Limited
- Asim Kumar Manna, "Business Mathematics and Statistics", McGraw Hill Education Private Limited

Exercise:

Check your progress:

1. Which of the following is a correct set?

   (a) A collection of most dangerous animals of the world.

   (b) A team of 5 best footballer of the world.

   (c) The collection of ten most interesting books.

   (d) The collection of all the days of a week beginning with the letter S.

2. The builder form of the following set is $X = \{5, 10, 15, 20, 25\}$

   (a) $X = \{x : x = 5n, n \in \mathbb{N} \text{ and } 1 \le n \le 5\}$

   (b) $X = \{x : x = 5n, n \in \mathbb{N} \text{ and } 1 < n \le 5\}$

   (c) $X = \{x : x = 5n, n \in \mathbb{N} \text{ and } 1 \le n < 5\}$

   (d) $X = \{x : x = 5n, n \in \mathbb{N} \text{ and } 1 < n < 5\}$

3. The Roster form of the following set is $X = \{x : x = n^2, n \in \mathbb{N} \text{ and } 1 \le n < 6\}$

   (a) $X = \{1, 8, 27, 64, 125\}$

   (b) $X = \{1, 4, 9, 16, 25\}$

   (c) $X = \{1, 2, 3, 4, 5\}$

   (d) $X = \{1, 4, 9, 16, 25, 36\}$

4. The Roster form of the following set is $X = \{x : x = 2n, n \in \mathbb{N} \text{ and } 1 \le n \le 6\}$

   (a) $X = \{2, 4, 6, 8, 10\}$

   (b) $X = \{1, 4, 6, 8, 10, 12\}$

   (c) $X = \{2, 4, 6, 8, 10, 12\}$

   (d) $X = \{2, 4, 9, 16, 25, 36\}$

Notes

5. The builder form of the following set is $X=\{1,4,9,16,........,100\}$

(a) $X=\{x: x=2n, n\in \mathrm{N} \text{ and } 1\le n\le 10\}$

(b) $X=\{x: x=n^2, n\in \mathrm{N} \text{ and } 1\le n\le 10\}$

(c) $X=\{x: x=5n, n\in \mathrm{N} \text{ and } 1\le n\le 10\}$

(d) $X=\{x: x=n^2, n\in \mathrm{N} \text{ and } 1\le n\le 10\}$

6. The set of tallest students in a class is

(a) a null set

(b) a singleton sets

(c) a finite set

(d) not a well-defined set.

7. Which of the following is an equivalent set?

(a) $X=\{2,4,6,8,10\}, Y=\{3,6,9,12,15\}$

(b) $X=\{5,10,11,13,18\}, Y=\{6,8,11,12\}$

(c) $X=\{1,2,3,4\}, Y=\{a,b,c,d,e\}$

(d) $X=\{f,j,k,l\}, Y=\{a,e,i,o,u\}$

8. Which of the following is null set?

(a) Set of even prime number

(b) Set of odd natural number divisible by 4

(c) Set of odd prime number

(d) Set of whole number

9. Which of the following set are finite?

(a) $X=\{x: x=4n+1, n\in \mathrm{N}\}$

(b) $X=\{x: x \text{ is a natural number}\}$

(c) $X=\{x: x \text{ is a prime number}\}$

(d) $X=\{x: x^2=4, x\in \mathrm{N}\}$

10. If $U=\{1,2,3,4,5,6,7,8,9,10\}$ and $X=\{2,5,7,8,10\}$, then is

Notes

(a) $X' = \{1,3,4,6,9\}$

(b) $X' = \{2,3,4,7,9\}$

(c) $X' = \{2,3,5,8,10\}$

(d) $X' = \{2,5,7,8,10\}$

11. If X and Y are two given sets, then $X \cap (X \cap Y)'$ is equal to
    (a) X
    (b) Y
    (c) ϕ
    (d) $X \cap Y$
12. If X and Y are two non-empty sets, then $P(X) \cup P(Y)$
    (a) $P(X \cup Y)$
    (b) $P(X \cap Y)$
    (c) $P(X) = P(Y)$
    (d) None of these
13. If X = {6} which of the following statement is correct?
    (a) $X = 6$
    (b) $6 \subset X$
    (c) $\{6\} \in X$
    (d) $6 \in X$
14. If X = {a, b, c, d} which of the following is not a subset of X?
    (a) {a, b}
    (b) {b, c}
    (c) {c, e}
    (d) {a, d}
15. If $n(X \cap Y) = 13, n(X) = 20, n(Y) = 44$ and $n(X \cup Y)$ is
    (a) 27
    (b) 13
    (c) 75
    (d) 51
16. If $n(X) = 6, n(Y) = 8, n(X \cup Y) = 12$, then $n(X \cap Y)$ is
    (a) 6

(b) 2

(c) 8

(d) 12

17. The number of all possible subsets of the set {1, 2, b, c, 5} is

(a) 5

(b) 10

(c) 16

(d) 32

18. If X = {b, c}, then the power set of X is

(a) $\{b, c\}$

(b) $\{\phi, \{b\}, \{c\}\}$

(c) $\{a, c\}$

(d) $\{\phi, \{b\}, \{c\}, \{b, c\}\}$

19. The total number of elements in a power set of X containing n elements is

(a) 2^n

(b) n^2

(c) $2^n - 1$

(d) 2^{2n}

20. Directions from Q20 to Q26: If X = {1, 2, 3}, Y = {2, 3, 4}, Z = {4, 5, 6, 7} Z ∩ Y = ?

(a) {2}

(b) (4)

(c) {2, 4}

(d) {5, 6}

21. X ∪ Y = ?

(a) {1, 4}

(b) {2, 3}

(c) {1, 2, 3, 4}

(d) {1, 2}

22. X ∩ Y = ?

(a) {1, 2, 3, 4}

(b) {2, 3}

Notes

(c) {1, 4}

(d) {4, 5}

23. $(X \cup Y) \cup Z = ?$

(a) {1, 2, 3, 4}

(b) {1, 2, 3, 4, 5, 6, 7}

(c) {2, 3, 4, 5, 6,7}

(d) {1, 2, 3, 4, 5}

24. $(X \cap Y) \cap Z = ?$

(a) {2, 3}

(b) {4}

(c) {1, 2, 3, 4, 5, 6,7}

(d) ϕ

25. $(X \cap Y) = ?$

(a) {1, 2, 3, 4}

(b) {2, 3}

(c) {1, 4, 5, 6,7}

(d) {4, 5, 6, 7}

26. $(X \cup Y)' = ?$

(a) {1, 2, 3, 4}

(b) {1, 4, 5, 6, 7}

(c) {2, 3, 4, 5, 6}

(d) {5, 6, 7}

27. The set $(X \cap Y')' \cup (Y \cap Z)$ is equal to

(a) $X' \cup Y \cup Z$

(b) $X' \cup Y$

(c) $X' \cup Z'$

(d) $X' \cap Y$

28. If X and Y are two sets, then $X \cap (Y \cup Z)$ equals

(a) X

(b) Y

(c) ϕ

(d) $X \cap Y$

Notes

29. Let X = {x : x is a positive in multiple of 3 less than 100} and Y = {y : y is a a prime number less than 20}. Then n (x) + n (Y) is

(a) 41

(b) 31

(c) 33

(d) 30

3. In a class of 100 students, 30 students play crickets and 25 students play football, and 15 students play both the games. Find the number of students who play neither.

(a) 50

(b) 55

(c) 40

(d) 60

31. From 50 students taking examinations in Economics, Business Mathematics and Accountancy, each of the student has passed in at least one of the subject, 37 passed Economics, 24 passed Business Mathematics and 43 Accountancy. At most 19 passed Economics and Business mathematics, at most 29 Economics and Accountancy and 20 Business Mathematics and Accountancy. The largest possible number of students that could have passed all three examinations is

(a) 14

(b) 13

(c) 15

(d) 20

32. In a college of 300 students, every student reads 5 different types of magazines and every magazine is read by 60 students. The number of magazines is

(a) at least 30

(b) atmost20

(c) exactly 25

(d) None of these

33. In a town of 840 persons,450 persons read Tamil, 300 read English and 290 read neither Tamil nor English. Then, the number of persons who read both

(a) 210

(b) 290

(c) 180

(d) 260

Notes

34. In a committee, 100 people speak Hindi, 40 speak English, and 20 speak both Hindi and English. The number of people speak at least one of these two languages is

(a) 150

(b) 60

(c) 120

(d) 135

35. Out of 800 children in a school, 224 played Badminton, 240 played Hockey, and 336 played Basketball. Out of this, 64 played both Basketball and Hockey, 80 played Basketball and Badminton and 40 played Badminton and Hockey, 24 played all the three games. The number of children who did not play any game is

(a) 128

(b) 216

(c) 240

(d) 160

36. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then A is

(a) Diagonal matrix

(b) Scalar matrix

(c) Unit matrix

(d) None of these

37. A column matrix has only

(a) One element

(b) One row with one or more columns

(c) One column with one or more rows

(d) One row with one element

38. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 6 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 3 \\ 3 & 0 & 5 \\ 5 & 7 & 0 \end{bmatrix}$ then $2A - 3B$ is

Notes

(a) $\begin{bmatrix} 8 & 4 & 15 \\ 9 & 10 & 29 \\ 27 & 37 & 18 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & -4 & 3 \\ 9 & -10 & 1 \\ 3 & 5 & -18 \end{bmatrix}$

(c) $\begin{bmatrix} -4 & 4 & -3 \\ -9 & 10 & -1 \\ -3 & -5 & 18 \end{bmatrix}$

(d) None of these

39. If $\begin{bmatrix} -1 & a+2 \\ a+1 & -2 \end{bmatrix} = \begin{bmatrix} a-3 & 4 \\ 3 & -a \end{bmatrix}$ then a = ?

(a) 1

(b) 2

(c) -2

(d) -1

40. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 12 \end{bmatrix}$, then

(a) AB = 0, BA = 0

(b) AB = 0, BA $\neq$ 0

(c) AB $\neq$ 0, BA = 0

(d) AB $\neq$ 0, BA $\neq$ 0

41. If $\begin{bmatrix} x+4 & 2x \\ 6 & y-1 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 6 & 2 \end{bmatrix}$, then

(a) x = 8, y = 3

(b) x = 4, y = –3

(c) x = 4, y = –1

(d) x = 4, y = 3

Notes

42. If $A=\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, then A^2 = ?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

43. If $X+Y=\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X-Y=\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then X and Y are

(a) $X=\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}, Y=\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(b) $X=\begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}, Y=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

(c) $X=\begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix}, Y=\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

(d) $X=\begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}, Y=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

44. If $A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, then

(a) $A^2 = B^2 = I$

(b) $A^2 = B^2 = -I$

(c) $A^2 = I$, $B^2 = -I$

(d) $A^2 = -I$, $B^2 = I$

Notes

45. If $A=\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}$, then $(A-2I)(A-3I)=?$

    (a) A

    (b) I

    (c) O

    (d) 5I

46. If $A=\begin{bmatrix}1 & 2 & 1\\ 3 & 4 & 2\\ 1 & 3 & 2\end{bmatrix}$ and $B=\begin{bmatrix}10 & -4 & -1\\ -11 & 5 & 0\\ 9 & -5 & 1\end{bmatrix}$ and, then

    (a) AB = BA

    (b) AB $\neq$ BA

    (c) AB = 2BA

    (d) None of these

47. If $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$, then $A^2-5A+6I=?$

    (a) $\begin{bmatrix}1 & -1 & -5\\ -1 & -1 & 4\\ -3 & -10 & 4\end{bmatrix}$

    (b) $\begin{bmatrix}1 & -1 & -3\\ -1 & -1 & -10\\ -5 & 4 & 4\end{bmatrix}$

    (c) O

    (d) I

48. A is a square matrix, then A + A' is

    (a) Unit matrix

    (b) Null matrix

    (c) Symmetric matrix

    (d) None of these

49. If $I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ is identity matrix, then I^{-1} is

Notes

(a) O

(b) 3I

(c) I

(d) Not exist

50. The inverse of $A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$ is

(a) $\begin{bmatrix}-3 & 2\\ 2 & -1\end{bmatrix}$

(b) $\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}$

(c) $\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$

(d) $\begin{bmatrix}3 & -2\\ -2 & 1\end{bmatrix}$

51. If $A=\begin{bmatrix}2x & 0\\ x & x\end{bmatrix}$, and $A^{-1}=\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}$, then $x=?$

(a) 1

(b) 2

(c) $\frac{1}{2}$

(d) None of these

52. A necessary and sufficient condition for a square matrix A to posses inverse is that

(a) A = 0

(b) *adj* A = 0

(c) |A| ≠ 0

(d) |A| = 0

53. The rank of the identity or unit matrix of order n is

(a) n – 1

(b) n

(c) n + 1

(d) n^2

54. If A is a non-singular matrix of order n, then the rank of A is

(a) 0

(b) 2

(c) n – 1

(d) n

55. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ is

(a) 0

(b) 1

(c) 2

(d) 3

56. If $A = \begin{bmatrix} 1 & 1 & 1 \\ a & a & a \\ a^3 & a^3 & a^3 \end{bmatrix}$, then rank of matrix A is

(a) 1

(b) 2

(c) 3

(d) 4

57. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$, then rank of matrix A is

(a) 0

(b) 1

(c) 2

(d) 3

58. The rank of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 5 \\ 1 & 2 & 4 \end{bmatrix}$ is

(a) 0

(b) 1

(c) 2

(d) 3

59. If $A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ -4 & 0 & -1 \end{bmatrix}$ and $M = \mathrm{I}_3 - A$, then the rank of M is

(a) 0

(b) 1

(c) 2

(d) 3

60. Rank of the matrix A, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ is

(a) 2

Notes

(b) 3

(c) 1

(d) 0

61. If $A = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 2 & 1 \end{bmatrix}$, then rank of matrix AB is

(a) 0

(b) 1

(c) 2

(d) 3

62. Which of the following is rank of matrix AA' ?

(a) rank A

(b) rank A'

(c) 1

(d) None of these

Answer Keys (Exercise):

Question	Answer	Question	Answer	Question	Answer
1	d	2	a	3	b
4	c	5	b	6	d
7	a	8	b	9	d
10	a	11	d	12	d
13	d	14	c	15	d
16	b	17	d	18	d
19	a	20	b	21	c
22	a	23	b	24	d
25	c	26	d	27	b
28	a	29	a	30	d
31	a	32	c	33	b
34	c	35	d	36	a
37	c	38	c	39	b
40	b	41	d	42	b
43	a	44	b	45	c
46	b	47	b	48	c
49	c	50	a	51	c
52	c	53	b	54	d
55	d	56	a	57	c
58	c	59	b	60	a
61	b	62	a		

Module – 2: Mathematical Logic

Notes

Course Contents:

- Basic concepts
- Propositions or Statements
- Truth Table
- Connectives and Compound Propositions
- Implication
- Bi-conditional of Connectives
- Converse
- Inverse and Contra positives of an Implication
- Tautology
- Logical equivalence
- Switching Circuits

Key Learning Objectives:

At the end of this block, you will be able to:

1. Define Propositions or Statements
2. Define Truth table
3. Define Connectives and Compound Propositions
4. Define Implication
5. Define Bi-conditional of Connectives
6. Define Converse, Inverse and Contra positives of an Implication
7. Define Tautology
8. Define Logical Equivalence
9. Define Switching Circuits

Structure:

Unit 2.1: Basic Concepts of Mathematical Logic

2.1.1 Introduction

2.1.2 Propositions or Statements

2.1.3 Compound Proposition

2.1.4. Connectives

2.1.4.1 Negation

2.1.4.2 Conjunction

Notes

Unit - 2.1: Basic Concepts of Mathematical Logic

Notes

Unit Outcome:

At the end of this unit, you will learn to

- Define Proposition or statements
- Construct Truth tables
- Define Connectives and Compound Propositions
- Define Implication, Negation and Bi-conditional of connectives
- Define Converse, Inverse and Contrapositive
- Define Algebra of Propositions
- Define Tautology, Contradiction and Contingency

2.1.1 Introduction

A statement, in practice, is constructed by means of words. We know that a word has more than one meaning, so there is a possibility of interpreting a group of words in more than one way and thereby creating confusion in the meaning of a statement. We use symbolic language to express mathematical statements and analysis of this symbolic language is the logic. Study of logic is very important in discrete mathematics as it provides the theoretical basis for many areas of computer science as artificial intelligence, digital logic, design, etc. In this unit, we will discuss the proposition or statements, compound statements, connectives, negation, conjunction, disjunction, truth table, implication, converse, inverse, contrapositive, bi-conditional, negation of compound statements, logical equivalence, tautology, contradiction and contingency also.

2.1.2 Proposition or Statement

A proposition or statement is a declarative (assertive) sentence that is either true or false, but not both simultaneously.

For example, “One plus two equals five” and “ One plus two equals three” are both statements, the first because it is false and the second because it is true.

The truth or falsity of a statement is called its truth value. Since only two possible truth values are admitted, this logic is sometimes called two-valued logic.

Since only two possible truth values are admitted, this logic is sometimes called two-valued logic.

Questions, exclamations and commands are not propositions.

The simple propositions are represented by the letters p, q, r, etc., which are known as propositions variables. Propositional variables can assume only two values - ‘true’ denoted by T or 1 and ‘false’ denoted by F or 0.

Notes

2.1.3 Compound Proposition

A proposition consisting of only a single propositional variable or a single propositional constant is called an atomic (primary or primitive) proposition or simply proposition, i.e., they can not be further subdivided.

A proposition obtained from the combinations of two or more propositions by means of logical operators or connectives of two or more propositions, or by negating a single proposition, is referred to as composite or compound proposition.

2.1.4 Connectives

The words and phrases (or symbols) used to form compound propositions are called connectives. There are five basic connectives called Negation, Conjunction, Disjunction, Conditional and Biconditional.

The following Symbols are used to represent connectives.

Symbol used	Connective word	Nature of the compound statement formed by the connective	Symbolic form	Negation
~, ¬, N	Not	Negation	~p	~(p) = p
$\wedge$	And	Conjunction	$p \wedge q$	$(\sim p) \wedge (\sim q)$
$\vee$	Or	Disjunction	$p \vee q$	$(\sim p) \vee (\sim q)$
$\rightarrow$	if……..then Conditional	conditional	$p \rightarrow q$	$p \wedge (\sim q)$
$\leftrightarrow$	if and only if	Bi-conditional	$p \leftrightarrow q$	$[p\ (\sim q)] \vee [q\ ()\ (\sim p)]$

Table 2.1.1 Table for different connective words

2.1.4.1 Negation

If p is any proposition, the negation of p, denoted by ~p and read as not p, is a proposition which is false when p is true and true when p is false.

Consider the statement

p : Dehradun is in Uttarakhand,

and negation of the above sentence is

~*p* : Dehradun is not in Uttarakhand,

Here 'not' is not a connective. Since it does not join two statements and is not really a compound statement.

Now, we take another statement

q : No student is intelligent.

and negation of the above sentence is

~q : Some students are intelligent.

Notes

2.1.4.2 Conjunction

If p and q are two statements, then the conjunction of p and q is the compound statement denoted by $p \wedge q$ and read as " p and q". The compound statement $p \wedge q$ is true when both p and q are true, otherwise, it is false.

2.1.4.3 Disjunction

If p and q are two statements, then disjunction of p and q is the compound statement denoted by $p \vee q$ and read as " p or q". The compound statement $p \vee q$ is true if at least one of p or q is true. If p and q are false, then it will be false.

2.1.4.4 Conditional Proposition or Implication

If p and q are proposition, the compound proposition "if p the q" denoted by $p \rightarrow q$ is called a conditional connective. The proposition p is called antecedent or hypothesis, and the proposition q is called the consequent or conclusion.

$p \rightarrow q$ is false under the condition that p is true and q is false.

Examples 2.1.1

1. If tomorrow is Wednesday, then today is Tuesday.
2. If it rains, then I will carry an umbrella.

Here p : Tomorrow is Wednesday is antecedent

q : Today is Tuesday is consequent

and p : It rains is antecedent

q: I will carry an umbrella is consequent

The connective if…….then can also be read as follows:

1. p is sufficient for q.
2. p only if q.
3. q is necessary for p.
4. q if p.
5. q follows from p.
6. q is consequence of p.

The truth table of $p \rightarrow q$ is given follows:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 2.1.2 Truth table for $p \rightarrow q$

Notes

2.1.4.5 Bi-conditional

If p and q are statements, then the compound statement p if and only if and q, denoted by $p \leftrightarrow q$ is called a biconditional statement and the connective if and only if is the biconditional connective. The biconditional statement $p \leftrightarrow q$ can also be stated as "p is a necessary and sufficient condition for q."

Example 2.1.2

i. He swims if and only if the water is warm.

ii. Sales of houses fall if and only if the interest rate rises.

The truth table for $p \leftrightarrow q$ is given in the following table. The statement $p \leftrightarrow q$ will be true only when both p and q are false.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 2.1.3 Truth table for $p \leftrightarrow q$

2.1.5 Truth Tables

A truth tables is a table that shows the truth value of a compound proposition for all possible cases.

For example, consider the conjunction of any two propositions p and q. The compound statement $p \wedge q$ is true when both p and q are true, otherwise it is false. There are four possible cases.

1. p is true and q is true.
2. p is true and q is false.
3. p is false and q is true.
4. p is false and q is false.

These four cases are listed in the first two columns and the truth values of $p \wedge q$ are shown in the third column of table.2.1.4 The truth tables for the other two connectives disjunction and negation of p are shown in table 2.1.5 and 2.1.6.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2.1.4 The truth table for $p \wedge q$

Notes

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 2.1.5 The truth table for $p \vee q$

p	~ p
T	F
F	T

Table 2.1.6 Truth table for ~p

2.1.6 Converse, Contrapositive and Inverse

If p and q are two propositions, then some other conditional propositions related to $p \to q$ are

a. Converse: The converse of $p \to q$ is $q \to p$.

b. Contrapositive : The contrapositive of $p \to q$ is $\sim q \to \sim p$.

c. Inverse: The inverse of $p \to q$ is $\sim p \to \sim q$.

The truth table of the four propositions follow:

p	q	Conditional $p \to q$	Converse $q \to p$	Inverse $\sim p \to \sim q$	Contrapositive $\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Table 2.1.7 Truth table for Conditional, Converse, Inverse and Contrapositive

Consider the statement

p: It rains

q: the crops will grow

The conditional proposition $p \to q$ states that,

$p \to q$: If it rains then the crops will grow.

The converse of $p \to q$ namely $q \to p$ states that,

$q \to p$: If the crops grow, then there has been rain.

The contrapositive of $p \to q$, namely $\sim q \to \sim p$ states that,

Notes

$\sim q \rightarrow \sim p$: If the crops do not grow then there has been no rain.

The inverse of $p \rightarrow q$, namely $\sim p \rightarrow \sim q$ states that,

$\sim p \rightarrow \sim q$: If it does not rain then the crops will not grow.

2.1.7 Negation of Compound Statements

2.1.7.1 Negation of Conjunction

The negation of a conjunction $p \wedge q$ is the disjunction of the negation of p and the negation of q. Symbolically we can write

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

Truth table of negation of a conjunction is shown in the following table:

p	q	~p	~q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Table 2.1.8 Truth table for negation of conjunction

2.7.1.2 Negation of Disjunction

The negation of a disjunction $p \vee q$ is the conjunction of the negation of p and the negation of q. Symbolically, we can write

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

Truth table of negation of a disjunction is shown in the following table:

p	q	~p	~q	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Table 2.1.9 Truth table for negation of disjunction

2.1.7.3 Negation of a Negation

A negation of negation of a statement is the statement itself.

Symbolically, we can write

$\sim(\sim p) \equiv p$

2.1.7.4 Negation of a Conditional

Notes

If p and q are two statements, then

$\sim(p \rightarrow q) \equiv p \wedge \sim q$

The negation of a conditional statement is shown in the following truth table:

p	q	~q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$p \wedge \sim q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Table 2.1.10 Truth table for negation of conditional

2.1.7.5 Negation of a Bi-conditional

If p and q are two statements, then

$\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q \equiv \sim p \leftrightarrow q$

The negation of a bi-conditional statement is shown in the following truth table:

p	q	~p	~q	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$	$\sim p \leftrightarrow q$	$p \leftrightarrow \sim q$
T	T	F	F	T	F	F	F
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	T	F	F	F

Table 2.1.11 Truth table for negation of bi-conditional

2.1.8 Algebra of Proposition

Proposition satisfy various laws. These laws are useful to simplify expression. All the laws except involution law in pairs are called dual pairs. For each expression, one finds the dual by replacing all T by F and all F by T and Replacing all by and all by. These laws are given in the followings:

1. Idempotent law

b. $p \vee q \equiv p$

c. $p \wedge q \equiv p$

2. Associative law

c. $(p \vee q) \vee r \equiv p \vee (q \vee r)$

d. $(p \vee q) \vee r \equiv p \wedge (q \vee r)$

5. Commutative law

a. $p \wedge q \equiv q \wedge p$

b. $p \vee q \equiv q \vee p$

Notes

4. **Distributive law**
   e. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   f. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
5. **Identity law**
   f. $p \vee T \equiv T$
   g. $p \vee F \equiv p$
   h. $p \wedge T \equiv p$
   i. $p \wedge F \equiv F$
6. **Complement law**
   g. $p \vee \sim p \equiv T$
   h. $p \wedge \sim p \equiv F$
   i. $\sim T \equiv F$
   j. $\sim F \equiv T$
7. **Involution law**
   h. $\sim(\sim p) \equiv p$
9. **De Morgan's law**
   j. $\sim(p \vee q) \equiv \sim p \wedge \sim q$
   k. $\sim(p \wedge q) \equiv \sim p \vee \sim q$

2.1.9 Logical Equivalence

If two propositions P(p, q,…..) and Q(p, q,…..) where p, q,…... are propositional variables have the same truth values in every possible case or is a tautology, then the propositions are called tautology, then the propositions are called logically equivalent or simply equivalent and denoted by

$$P(p,q,......) \equiv Q(p,q,......)$$

or

$$P(p,q,......) \Leftrightarrow Q(p,q,......)$$

It is always permissible, and sometimes desirable to replace a given proposition by an equivalent one.

To test whether two propositions P and Q are logically equivalent the following steps are followed.

1. Construct the truth table for P.
2. Construct the truth for Q using the same propositional variables.
3. Check each possible combinations of truth values of the propositional variables to see whether the value of P is the same as the truth value of Q. If the truth value of P in each row is equal to the truth value of Q, then P and Q are logically equivalent.

2.1.10 Tautology

A statement pattern whose truth value is true for all possible combinations of the truth values of its prime components is called a tautology. We denoted tautology by t.

Statement pattern $P \vee \sim p$ is a tautology.

Notes

2.1.11 Contradiction

A statement pattern whose truth value is false for all possible combinations of the truth values of its prime components is called a contradiction. We denoted contradiction by c.

Statement pattern $P \wedge \sim p$ is a contradiction.

2.1.12 Contingency

A statement pattern which is neither a tautology nor a contradiction is called a contingency.

$p \wedge q$ is a contingency.

Solved Examples:

Ex-2.1 Form the conjunction of p and q for each of the following.

a. p : Ram is healthy. q : He has blue eyes.

b. p : It is cold. q : It is raining.

c. p : $5x + 6 = 26$ q : $x > 3$

Solution :

(a) $p \wedge q$: Ram is healthy and he has blue eyes.

(b) $p \wedge q$: It is cold and raining.

(c) $p \wedge q$: $5x + 6 = 26$ and $x > 3$

Ex-2.2 Assign a truth value to each of the following statements.

(a) $5 < 5 \vee 5 < 6$

(b) $5 \times 4 = 21 \vee 9 + 7 = 17$

(c) $6 + 4 = 10 \vee 0 > 2$

Solution:

a. True, since one of its components, i.e., $5 < 6$ is true.

b. False, since both of its components are false.

c. True, since one of its components, i.e., $6 + 4 = 10$ is true.

Ex-2.3 If p : It is cold and q : It is raining.

Write simple verbal sentence which describes each of the following statements.

(a) $\sim p$

(b) $p \wedge q$

(c) $p \vee q$

(d) $p \vee \sim q$

Notes

Solution:

(a) ~p : It is not cold.

(b) p $\wedge$ q : It is cold and raining.

(c) p $\vee$ q : It is cold or raining.

(d) p$\vee$ ~q : It is cold or it is not raining.

Ex-2.4 Construct a truth table for ~(p $\vee$ q) $\vee$ (~p$\wedge$ ~q) .

Solution: The truth table for ~(p $\vee$ q) $\vee$ (~p$\wedge$ ~q) .

p	q	~p	~q	p $\vee$ q	~(p$\wedge$ ~q)	~p$\wedge$ ~q	~(p $\wedge$q) $\vee$ ~(p$\wedge$ ~q)
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T	T

Ex-2.5 Construct a truth table for p$\vee$ ~ q $\rightarrow$ p.

Solution: The truth table for p$\vee$ ~ q $\rightarrow$ p

p	q	~q	p$\wedge$ ~q	p$\vee$ ~q $\rightarrow$ p
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	F

Ex-2.6 Prove that if x^2 is divisible by 4, then x is even.

Solution: Let p and q be the propositions such that

p : x^2 is divisible by 4

and q : x is even

the conditional is of the form p $\rightarrow$ q. The contrapositive is ~q $\rightarrow$ ~p, which states in words:

If x is odd, then x^2 is not divisible by 4.

The proof of contrapositive is easy.

Since x is odd, one ca write x = 2k + 1, for some integer k. Hence

$$x^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 4\left(k^2 + k + \frac{1}{4}\right)$$

Since k^2 + k is an integer, and $k^2 + k + \frac{1}{4}$ is not an integer, x^2 is not divisible by 4.

Ex-2.7 Write the negation of each of the following conjunctions:

(a) Paris is in France and London is in England.

(b) $2 + 4 = 6$ and $7 < 12$

Solution:

(a) Let p : Paris is in France and q : London is in England

Then the conjunction is given by $p \wedge q$.

Now

$\sim p$: Paris is not in France.

$\sim q$: London is not in England.

Therefore, negation of $p \wedge q$ is given by

$\sim (p \wedge q)$: Paris in not in France or London is not in England.

Let p : $2 + 4 = 6$ and q : $7 < 12$

Then, the conjunction is given by $p \wedge q$.

Now,

$\sim p$: $2 + 4 \neq$ and $\sim q$: $7 \nless 12$

The negation of $p \wedge q$ is given by

$\sim (p \wedge q)$: $2 + 4 = 6$ and $\sim q$: $7 \nless 12$

Ex-2.8 Verify negation of a negation for the statement

p : Roses are red

Solution: The negation of p is given by

$\sim p$: Roses are not red.

Hence, the negation of negation of p is $\sim(\sim p)$:

It is not the case that roses are not red.

or

It is false that roses are not red.

or

Roses are red.

Ex-2.9 Write the negation of the statements "If it is raining, then the game is cancelled".

Solution: Let p : It is raining, and q : The game is cancelled.

The given statement can be written as $p \rightarrow q$. The negation of $p \rightarrow q$ is written as

$\sim(p \rightarrow q) \equiv p \wedge \sim q$

Notes

Notes

Hence, the negation of the given statement is it is raining and the game is not cancelled.

Ex-2.10 Using the truth table prove the following logical equivalence

$p \rightarrow (q \rightarrow p) \equiv \sim p \rightarrow (p \rightarrow q)$

Solution: The truth table for $p \rightarrow (q \rightarrow p) \equiv \sim p \rightarrow (p \rightarrow q)$

p	q	$q \rightarrow p$	$p \rightarrow (q \rightarrow q)$	$\sim p$	$p \rightarrow q$	$\sim p \rightarrow (p \rightarrow q)$
T	T	T	T	F	T	T
T	F	T	T	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

So, from the column 4 and 7,

$p \rightarrow (q \rightarrow p) \equiv \sim p \rightarrow (p \rightarrow q)$

Ex-2.11 Using the truth table prove the following logical equivalence

$p \rightarrow (q \wedge r) \equiv (p \wedge q)\ (p \rightarrow r)$

Solution: The truth table for $p \rightarrow (q \wedge r) \equiv (p \wedge q)\ (p \rightarrow r)$

p	q	R	$q \wedge r$	$p \rightarrow (q \wedge r)$	$(p \wedge q)$	$(p \rightarrow r)$	$p \rightarrow (q \wedge r) \equiv (p \wedge q)\ (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

So, from the column 5 and 8,

$p \rightarrow (q \wedge r) \equiv (p \wedge q)\ (p \rightarrow r)$

Ex-2.12 Examine whether the following statement pattern is a tautology or a contradiction or a contingency.

$(p \wedge q) \rightarrow (q \vee p)$

Solution: The truth table for $(p \wedge q) \rightarrow (q \vee p)$

P	Q	$p \wedge q$	$q \vee p$	$(p \wedge q) \rightarrow (q \vee p)$
T	T	T	T	T
T	F	F	T	T
T	T	F	T	T
T	F	F	F	T

All the values in the last column of the above truth table are T.

So, $(p \wedge q) \rightarrow (q \vee p)$ is a tautology.

Ex- 2.13 Examine whether the following statement pattern is a tautology or a contradiction or a contingency.

Solution: The truth table for $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$

p	q	~p	$q \rightarrow p$	$\sim q \vee p$	$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

All the values in the last column of the above truth table are T.

So, $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ is a tautology.

Ex- 2.14 Examine whether the following statement pattern is a tautology or a contradiction or a contingency.

Solution: The truth table for $[(p \rightarrow q) \wedge q] \rightarrow p$

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$[(p \rightarrow q) \wedge q] \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

The entries in the last column of the above truth table are neither all T nor all F.

Hence $[(p \rightarrow q) \wedge q] \rightarrow p$ is a contingency.

Ex-2.15 Examine whether the following statement pattern is a tautology or a contradiction or a contingency.

Solution: The truth table for $(p \leftrightarrow q) \wedge (p \rightarrow \sim q)$

p	q	~q	$p \leftrightarrow q$	$p \rightarrow q$ ~q	$(\pi \leftrightarrow \theta) \wedge (\pi \rightarrow \sim\theta)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	F	T	F
F	F	T	T	T	T

The entries in the last column of the above truth table are neither all T nor all F.

Hence $(p \leftrightarrow q) \wedge (p \rightarrow \sim q)$ is a contingency.

Notes

Ex-2.16 Examine whether the following statement pattern is a tautology or a contradiction or a contingency. [p→ (~qvr)] ↔ ~[p → (q→r)]

Solution: The truth table for [p→ (~qvr)] ↔ ~[p → (q→r)]

p	q	r	~q	~qvr	p → (~qvr)	q → r	p → (~qvr)	[p→ (~qvr)] ↔ ~[p → (q→r)]
T	T	T	F	T	T	T	T	F
T	T	F	F	F	F	F	F	T
T	F	T	T	T	T	T	T	F
T	F	F	T	T	T	T	T	F
F	T	T	F	T	T	T	T	F
F	T	F	F	F	T	F	T	F
F	F	T	T	T	T	T	T	F
F	F	F	T	T	T	T	T	F

All the values in the last column of the above truth table are F.

So, [p→ (~qvr)] ↔ ~[p → (q→r)] is a contradiction.

Summary:

- A proposition or statement is a declarative (assertive) sentence that is either true or false, but not both simultaneously.
- A proposition obtained from the combinations of two or more propositions by means of logical operators or connectives of two or more propositions or by negating a single proposition is referred to composite or compound proposition.
- The words and phrases (or symbols) used to form compound propositions are called connectives. There are five basic connectives called Negation, Conjunction, Disjunction, Conditional and Biconditional.
- If p is any proposition, the negation of p, denoted by ~p and read as not p, is a proposition which is false when p is true and true when p is false.
- If p and q are two statements, then conjunction of p and q is the compound statement denoted by $p \vee q$ and read as “p and q”.
- If p and q are proposition, the compound proposition “if p the q” denoted by is called a conditional connective.
- If p and q are two statements, then disjunction of p and q is the compound statement denoted by $p \vee q$ and read as “p or q”.
- If p and q are proposition, the compound proposition “if p then q” denoted by $p \to q$ is called a conditional connective.
- A truth table is a table that shows the truth value of a compound proposition for all possible cases.
- If p and q are two propositions, then some other conditional propositions related to $p \to q$ are

- **Converse:** The converse of $p \rightarrow q$ is $q \rightarrow p$.
- **Contrapositive:** The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.
- **Inverse:** The inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.
- If two propositions P(p, q,…..) and Q(p, q,…..) where p, q,….. are propositional variables have the same truth values in every possible case or $P \leftrightarrow Q$ is a tautology, then the propositions are called tautology, then the propositions are called logically equivalent.
- A statement pattern whose truth value is true for all possible combinations of the truth values of its prime components is called a tautology.
- A statement pattern whose truth value is false for all possible combinations of the truth values of its prime components is called a contradiction.
- A statement pattern which is neither a tautology nor a contradiction is called a contingency.

Activity:

1. Construct a truth table for the statement pattern .
2. Use the truth table to prove the distributive law $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
3. What will be the negation of the statement "If he studies, he will pass the examination".
4. What will be the negation of the statement "The computer program is correct if and only if, it produces the correct answer for all possible sets of input data".

Notes

Unit - 2.2: Switching Circuits

Recall Session:

In the previous unit, you studied about:

- Proposition or statements
- Truth tables
- Connectives and Compound Propositions
- Implication, Negation and Bi-conditional of connectives
- Converse, Inverse and Contrapositive
- Algebra of Propositions
- Tautology, Contradiction and Contingency

Unit Outcome:

At the end of this unit you will learn to

1. Define Switch circuits

2.2.1 Introduction

In the previous unit, we learned about the proposition or statements, compound statements, connectives, negation, conjunction, disjunction, truth table, implication, converse, inverse, contrapositive, bi-conditional, negation of compound statements, logical equivalence, tautology, contradiction and contingency also. In this unit, we will learn about switch circuits.

2.2.2 Switching circuit

A switch is a two state devices used to control the flow of current in a circuit.

We shall denote the switches by letters S, S1, S2,…… etc.

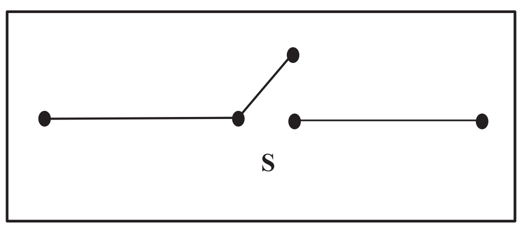


Fig. 2.1.1 Pictorial representation of Switch circuit

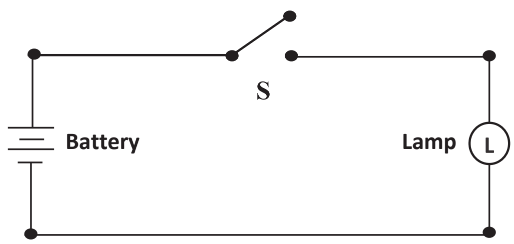


Fig. 2.1.2 Switch circuit

Here, we consider a circuit containing an lamp, controlled by a switch S.

When switch is closed (on or 1), then current flows in the circuit and hence the lamp glows. When the switch S is open (off or 0), then current does not flow in the circuit and subsequently the lamp does not glow.

The theory of symbolic logic can be used to represent a circuit by a statement pattern. Conversely, for given statement patterns we can construct a circuit.

Switches having the same state can be represented by the same letter and called equivalent switches and the switches having opposite states are represented by S and and these switches are called complementary switches.

In the figure, switch S1 corresponds to statement letter p in the corresponding statement pattern.

We can write it as p: switch S_1 and ~p : switch S'_1 .

The correspondence between switch S_2 corresponds to statement letter q in the corresponding statement pattern.

We can write it as q: switch S_2 and ~q : switch S'_2 .

We consider all possible combinations of states of all switches in the circuit and prepare a table, called "Input Output table", which is similar to truth table of the corresponding statement pattern.

Two switches in series

Two switches S_1 and S_2 connected in series and electric lamp 'L' as shown in fig. 2.2.3

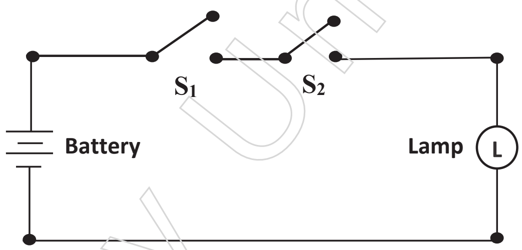


Fig. 2.2.3 Two switches in series

Let p: The switch S_1

q: The switch S_2

L: The lamp L.

Input output table (switching table) for p ∧ q.

p	q	p ∧ q
1	1	1
1	0	0
0	1	0
0	0	0

Table 2.2.1 Input output table for two switches in series

Notes

Two switches in parallel

Two switches S_1 and S_2 connected in parallel and electric lamp 'L' as shown in fig. 2.2.4

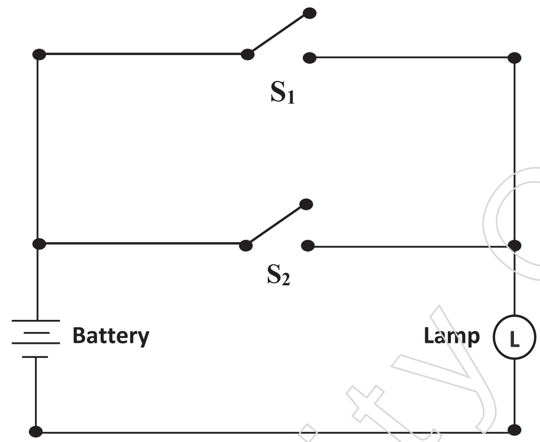


Fig. 2.2.4 Input output table for two switches in parallel

Let p: The switch S1

q: The switch S2

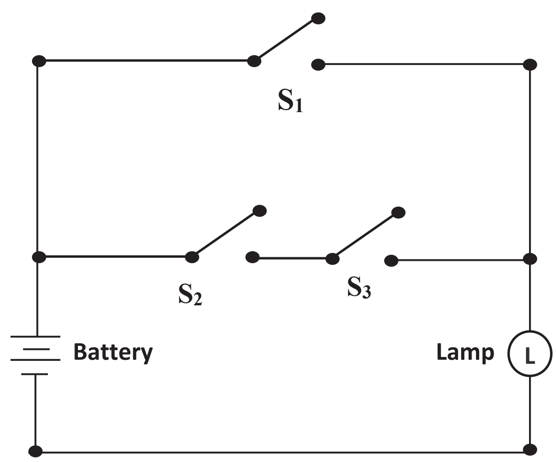
L: The lamp L

Input output table (switching table) for p ∨ q.

p	q	p ∨ q
1	1	1
1	0	1
0	1	1
0	0	0

Table. 2.2.2 Truth table for two switches in parallel

Ex- 2.17 Express the following circuit in the symbolic form of logic and write the input-output table.



Solution: Let p : the switch S1 is closed

q : the switch S2 is closed

r: the switch S3 is closed

~p : the switch S1 is open

~q : the switch S2 is open

~r : the switch S3 is open

l : the lamp L is on

The symbolic form of the given circuit is : $p \vee (q \wedge r) \equiv l$

Ex-2.18 Construct the switching circuit of the following:

$(\sim p \wedge q) \vee (p \wedge \sim r)$

Solution: Let p : the switch S_1 is closed

q : the switch S_2 is closed

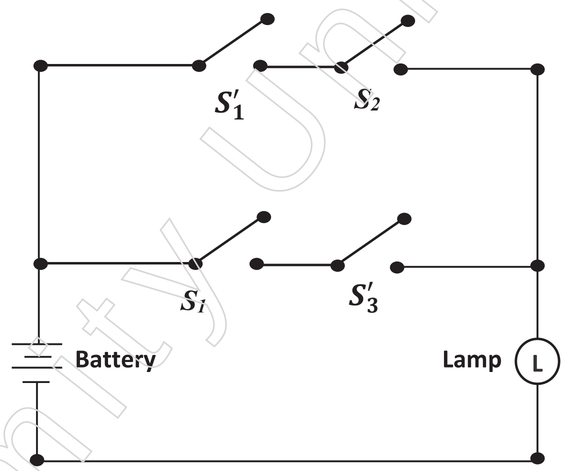
r: the switch S_3 is closed

~p : the switch S'_1 is closed or the switch S1 is open

~q : the switch S'_2 is closed or the switch S2 is open

~r : the switch S'_3 is closed or the switch S3 is open

Then the switching circuit corresponding to the given statement pattern is:



Summary:

- A switch is a two state device used to control the flow of current in a circuit.
- Switches having the same state can be represented by the same letter and called equivalent switches.
- The switches having opposite states are represented by S and S' and these switches are called complementary switches.

Notes

Further Readings:

1. Dr. Swapan kumar sarkar, “A Textbook of Discrete Mathematics”, S Chand And Company Limited, Ramnagar New Delhi.

2. Dr. R. K. Rajput, “Mathematics For BCA”, Discovery Publishing House PVT. Ltd. , New delhi.

Exercise:

Check your progress:

1. Which of the following is a statement
   (a) Open the door
   (b) Do your homework
   (c) Switch on the fan
   (d) Two plus two is four
2. Which of the following is a statement
   (a) May you live long!
   (b) May God bless you!
   (c) The sun is a star
   (d) Hurrah! we have won the match
3. Which of the following is not a statement
   (a) Every set is a finite set
   (b) 8 is less than 6
   (c) Where are you going?
   (d) The sum of all interior angles of a triangle is 180 degree
4. Which of the following is not a statement
   (a) Please do me a favour
   (b) 2 is an even integer
   (c) 2 + 1 = 3
   (d) The number 17 is prime
5. Let P, Q and R be three atomic prepositional declarations. Let X denote $(p \vee Q) \rightarrow R$, and Y denote $(p \rightarrow R) \vee (Q \rightarrow R)$. Which of the following is a tautology?
   (a) $X \equiv Y$
   (b) $X \rightarrow Y$
   (c) $Y \rightarrow X$
   (d) $\neg Y \rightarrow X$
6. Consider the following propositional statements:

   P_1 : $[(A \wedge B) \rightarrow C] \equiv [(A \rightarrow C) \wedge (B \rightarrow C)]$

P_2 : $[(A \vee B) \rightarrow C] \equiv [(A \rightarrow C) \vee (B \rightarrow C)]$

Which one of the following is true?

(a) P_1 is tautology, but not P_2

(b) P_2 is tautology, but not P1

(c) P_1 and P_2 are both tautologies

(d) Both P_1 and P_2 are not tautologies

7. Negation of the conditional statement "If it rains, I shall go to school" is

(a) It rains and I shall go to school

(b) It rains and I shall not go to school

(c) It does not rains and I shall go to school

(d) None of these

8. Which of the following is a contradiction

(a) $(p \wedge q) \wedge \sim(p \vee q)$

(b) $p \vee (\sim p \wedge q)$

(c) $(p \Rightarrow q) \Rightarrow p$

(d) None of these

9. $\sim(\sim p) \wedge q$ is equal to

(a) $\sim p \vee q$

(b) $p \wedge q$

(c) $p \wedge \sim q$

(d) $\sim p \wedge \sim q$

10. If p, q, r are simple propositions, then $(p \wedge q) \wedge (q \wedge r)$ is true then

(a) p, q, r are all false

(b) p, q, r are all true

(c) p, q are all true and r is false

(d) p is true and q and r are false

11. The compound proposition p and q are called logically equivalent if ______ is tautology.

(a) $p \leftrightarrow q$

(b) $p \rightarrow q$

(c) $\neg(p \vee q)$

(d) $(p \vee \neg q)$

12. $p \rightarrow q$ is logically equivalent to

(a) $\sim p \vee \sim q$

(b) $p \vee \sim q$

Notes

(c) $\sim p \vee q$

(d) $\sim p \wedge q$

13. $p \vee q$ is logically equivalent to

(a) $\neg q \to \neg p$

(b) q () p

(c) $\neg p \to \neg q$

(d) $\neg p \to q$

14. $\neg(p \leftrightarrow q)$ is logically equivalent to

(a) $q \leftrightarrow p$

(b) $p \leftrightarrow \neg q$

(c) $\neg p \leftrightarrow \neg q$

(d) $\neg q \leftrightarrow \neg p$

15. $p \wedge q$ is logically equivalent to

(a) $\neg(p \to \neg q)$

(b) $(p \to \neg q)$

(c) $(\neg p \to \neg q)$

(d) $(\neg p \to q)$

16. Which of the following statement is correct?

(a) $p \vee q \equiv q \vee p$

(b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

(c) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(d) All of mentioned

17. $p \leftrightarrow q$ is logically equivalent to

(a) $(p \to q) \to (q \to p)$

(b) $(p \to q) \vee (q \to p)$

(c) $(p \to q) \wedge (q \to p)$

(d) $(p \to q) \to (q \to p)$

18. $(p \to q) \vee (p \to r)$ is logically equivalent to

(a) $p \to (q \wedge r)$

(b) $p \to (q \vee r)$

(c) $p \wedge (q \vee r)$

(d) $p \vee (q \wedge r)$

19. $(p \to r) \vee (q \to r)$ is logically equivalent to

(a) $(q \wedge q) \vee r$

Notes

(b) $(p \vee q) \rightarrow r$

(c) $(p \wedge q) \rightarrow r$

(d) $(p \rightarrow q) \rightarrow r$

20. $\neg(p \leftrightarrow q)$ is logically equivalent to

(a) $p \leftrightarrow \neg q$

(b) $\neg p \leftrightarrow q$

(c) $\neg p \leftrightarrow \neg q$

(d) $\neg q \leftrightarrow \neg p$

Answer Keys (Exercise):

Question	Answer	Question	Answer	Question	Answer
1	d	2	c	3	c
4	a	5	b	6	b
7	b	8	a	9	b
10	b	11	a	12	c
13	d	14	b	15	a
16	d	17	c	18	a
19	c	20	a		

Notes

Module – 3: Group and Subgroup

Course Contents:

- Binary operations
- Properties of Binary Operations
- Semi-group
- Monoid
- Group
- Subgroups and other groups

Key Learning Objectives:

At the end of this block, you will be able to:

1. Define Binary Operations
2. Describe various types of Binary Operations
3. Define Group
4. Define Semi group and Monoid
5. Define Subgroups
6. Define Order of an element of a Group
7. Define Cyclic Group

Structure:

UNIT 3.1: Group

3.1.1 Introduction

3.1.2 Binary operations

3.1.3 Types of Binary Operations

3.1.4 Algebraic Structure

3.1.5 Group

3.1.6 Groupoid

3.1.7 Semi-group

3.1.8 Monoid

3.1.9 Quaternian Group

3.1.10 Abelian Group or Commutative Group

3.1.11 Finite and Infinite Groups

3.1.12 Order of a Group

3.1.13 General properties of a Group

Notes

Unit-3.2: Sub Group and Other Groups

Notes

Unit - 3.1: Group

Unit Outcome:

At the end of this unit, you will learn to

- Define Binary operations
- Describe various types of Binary operations
- Define Group, Groupoid, Semi-group, Monoid and Quaternian Group
- Define Abelian group, Finite and Infinite Groups and order of a group
- Define General properties of a group

3.1.1 Introduction

The theory of groups, an important part in the present mathematical scope, started early in the 19th century in connection with the solution of algebraic equations. This idea was later generalised to the concept of an abstract group. An abstract group is essentially a study of the set with an operation defined on it. Group theory has many applications in the internal and external fields of mathematics. The group originates in a number of apparently unrelated subjects. In fact, they also appear in crystallography and quantum mechanics, in geometry and topology, analysis and algebra, and even in biology. In this unit, before we start talking about a group, it would be fruitful to discuss binary operations on a set because there are elements on which a break operation can be built on its elements. We can get the third element of the set by combining the two elements of the set. It is not true always. That is why this concept needs more attention.

3.1.2 Binary Operations

A binary operation 'o' on G is a mapping from G × G to G, i.e., o : G × G → G where the image of (a, b) of G × G under 'o', i.e., o(a, b), is denoted by a o b.

3.1.3 Types of Binary Operations

1. **Commutative Operation:** A binary operation over a set G is said to be commutative, if, for every pair of the element a, b $\in$ G,

   a o b = b o a

   So, addition and multiplication are commutative binary operations for natural numbers whereas subtraction and division are not commutative because, for a, b $\in$ a, and a – b $\neq$ b – a and a $\div$ b $\neq$ b $\div$ a can not be true for every pair of natural number a and b.

   Example 3.1.1 $3 - 2 \neq 2 - 3$

   Example 3.1.2 $3 \div 2 \neq 2 - 3$

2. **Associative operation:** A binary operation o on set G is called associative if, a o (boc) = (a o b) o c for all a, b, c $\in$ G.

3. **Distributive operation:** Let o and o' be two binary operations defined on a set G. Then operation o' is said to be left distributive with respect to operation o if

   a o' (boc) = (a o' b) o (a o' c) for all a, b, c $\in$ G

   and is said to be right distributive with respect to o if,

   (b o c) o' a = (b o' a) o (c o' a) for all a,b,c $\in$ G

4. **Identity:** A composition o in a set G is said to admit of an identity if there exists an element for all e $\in$ G such that

   a o e = a = e o a $\forall$ a $\in$ G.

Where e is called an identity element and the algebraic structure (G, o) is said to have an identity element with respect to o.

Example 3.1.3 If a $\in$ R, the set of real numbers then 0 is an additive identity of R because

a + 0 = a = 0 + a $\forall$ a$\in$R

N, the set of natural numbers, has no identity element with respect to addition because 0$\notin$N .

Example 3.1.4 1 is the multiplicative identity of N as

a.1 = a = 1.a $\forall$ a$\in$N

1 is the identity of multiplication for I (set of integers), Q (set of rational numbers), R (set of real numbers).

5. **Inverse:** An element a $\in$ G is said to have its inverse with respect to certain operation o if there b $\in$ G such that

   a o b = e = b o a

   e being the identity in G with respect to o.

   Such an element b, usually denoted by a^{-1} is called the inverse of a. Thus

   a^{-1} o a = e = a o a^{-1} for a$\in$G

In the set of integers, the inverse of an integer a with respect to ordinary addition operation is -a and in the set of non-zero rational numbers, the inverse of a with respect to multiplication is $\frac{1}{a}$ which belongs to the set.

3.1.4 Algebraic Structure

A non-empty set G together with at least one binary operation defined on it is called an algebraic structure. Thus, if G is a non-empty set and 'o' is a binary operation on G, then (G, o) is an algebraic structure.

(N,+), (1,+), (1,–), (R,+,.)

are all algebraic structures. Thus addition and multiplication are both binary operations on the set R of real numbers, (R,+,.) is an algebraic structure equipped with two operations.

Notes

Example 3.1.5 If the binary operation o on Q the set of rational numbers is defined by a o b = a + b – ab, for a,b $\in$ Q.

Show that o is commutative and associative.

Solution (i) 'o' is commutative in Q because if a,b $\in$ Q, then

$$\begin{aligned} a \ o \ b &= a+b-ab \\ &= b+a-ba \\ &= b \ o \ a \end{aligned}$$

(ii) 'o' is associative in Q because if , then

$$\begin{aligned} a \ o \ (b \ o \ c) &= a \ o (b+c-bc) \\ &= a+(b+c-bc)-a(b+c-bc) \\ &= a+b-ab+c-(a+b-ab)c \\ &= (a \ o \ b) o \ c \end{aligned}$$

Example 3.1.6 Given that S = {A, B, C, D} where A=ϕ, B={a}, C={a,b} & D={a,b,c} show that S is closed under the binary operations $\cup$(union of sets) and $\cap$ (intersection of sets) on S.

Solution (i) A $\cup$ B = $\phi \cup$ {a} = {a} = B

Similarly, A $\cup$ C = C, A $\cup$ D = D and A $\cup$ A = A

Also, $B \cup B = B, \ B \cup C = \{a\} \cup \{a,b\} = \{a,b\} = C$

$$B \cup D = \{a\} \cup \{a,b,c\} = \{a,b,c\} = D$$

$$C \cup C = C, \ C \cup D = \{a,b\} \cup \{a,b,c\} = \{a,b,c\} = D$$

Hence $\cup$ is a binary operation on S.

(ii) Again, $A \cap A = A, \ A \cap B = \phi \cup \{a\} = \phi = A$

Similarly, A $\cap$ C = A, A $\cap$ D = A

Also, $B \cap B = B, \ B \cap C = \{a\} \cap \{a,b\} = \{a\} = B$

$$B \cap D = \{a\} \cap \{a,b,c\} = \{a\} = B$$

$$C \cap C = C, \ C \cap D = \{a,b\} \cap \{a,b,c\} = \{a,b\} = C$$

Hence $\cap$ is a binary operation on S.

3.1.5 Group

An algebraic structure (G, o) where g is a non-empty set with an operation 'o' defined on it is said to be a group if the operation satisfies the following axiom (called group axiom)

(G_1) **Closure axiom** G is closed under the operation o, i.e., a o b$\in$G, for all a,b$\in$G .

(G_2) **Associative axiom** the binary operation o is associative, i.e.,

Notes

$(a \circ b) \circ c = a \circ (b \circ c)\ \forall\ a,b,c \in G$

(G_3) **Identity axiom** There exists an element $e \in G$ such that

$e \circ a = a \circ e = a\ \forall\ a \in G$

The element e is called the identity of 'o' in G.

(G_4) **Inverse axiom** Each element of G posses inverse, i.e., for each element $a \in G$, there exists an element $b \in G$ such that

$a \circ b = e = b \circ a$

e being the identity in G with respect to o.

The element b is then called the inverse of a with respect to 'o' and we write $b=a^{-1}$. Thus a^{-1} is an element of G such that

$a \circ a^{-1} = e = a^{-1} \circ a$

3.1.6 Groupoid

If an algebraic system (G, o) satisfies only G_1 axiom, it is called a groupoid or a quasi-group.

The groupoid is a set G with a binary operation o defined on G such that G is closed under the operation o, that is, if $a, b \in G$, then $a \circ b \in G$.

Example: If I_0 is the set of odd integers, then the system (I_0, +) is not a groupoid, because the set I_0 is not closed under addition

$3 + 7 = 10 \notin I_0$

3.1.7 Semi group

If an algebraic system (G, o) satisfies only G_1 and G_2 axiom, it is called a semi group or a demi-group.

Example 3.1.7 If I is the set of integers then the system (I, +) and (I,.) are semi groups, since I is closed under addition as well as multiplication and for $a,b,c \in I$ it follows associative law, i.e.,

$(a + b) + c = a + (b + c)$ and $(a.b).c = a.(b.c)$

On the other hand, the system (I, -) is not a semi group, because subtraction does not satisfy the associative law, i.e.,

$(a - b) - c \neq a - (b - c)$, for $a,b,c \in I$

Similarly (N,.) and (R, +) are also semi-group.

3.1.8 Monoid

A semi-group which also satisfies G3 is called a monoid. Monoid is usually denoted by M.

Monoid is an associative groupoid with an identity element.

Notes

3.1.9 Quaternian Group

If the set A is given by

A = {±1, ±i, ±j, ±k}

and multiplication binary operation is defined as

$i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$,

then A is called quaternian group.

Example 3.1.8 $A = \left\{ \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \pm\begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \pm\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

is a quaternian group.

3.1.10 Abelian group

A group (G, o) is said to be abelian or commutative if the composition 'o' is commutative. i.e.,

$a \text{ o } b = b \text{ o } a \ \forall \ a,b \in G$

A group which is not abelian is called non-abelian.

Example 3.1.8 (Z, +), (Q, +), (R, +), (C, +), (Q_0, +), (Q^+, ×) and (R^+, ×) are the example of abelian group.

Example 3.1.9 Show that the set of all even integers (including zero) with additive property is an abelian group.

Solution The set of all even integers (including zero) is

I = {0, ±2, ±4, ±6,......}

Now we will discuss the group axioms one by one:

(G_1) The sum of two even integers is always an even integer, therefore **closure axiom** is satisfied.

(G_2) The addition is associative for even integers; hence **associative axiom** is satisfied.

(G_3) 0 ∈ **I**, which is an additive identity in **I**, hence **identity axiom** is satisfied.

(G_4) Inverse of an even integers a is the even integer –a in the set, so **axiom of inverse** is satisfied.

(G_5) **Commutative law** is also satisfied for addition of even integers. Hence the set forms an **abelian group**.

Example 3.1.10 Show that the set of all non-zero rational numbers with respect to binary operation of multiplication is a group.

Solution Let the given set be denoted by Q_0. Then by group axioms, we have

(G_1) The product of two non-zero rational numbers are also a non-zero rational number. Therefore, Q0 is closed with respect to multiplication. Hence, closure axiom is satisfied.

(G_2) For rational numbers

(a.b).c = a. (b.c) for all a, b, c $\in Q_0$

Hence, **associative axiom** is satisfied.

(G_3) Since 1, a multiplicative identity is a rational number, identity axiom is satisfied.

(G_4) If a $\in Q_0$, then obviously, $\frac{1}{a} \in Q_0$. Also

$$\frac{1}{a}.a = 1 = a.\frac{1}{a}$$

So that $\frac{1}{a}$ is the multiplicative inverse of a. Thus, inverse axiom is also satisfied.

Hence Q_0 is a group with respect to multiplication.

3.1.11 Finite and Infinite Groups

If a group has a finite number of distinct elements, it is called the finite group; otherwise, an infinite group.

3.1.12 Order of a Group

The number of elements or members in a finite group is called the order of the group. The order of an infinite group is always infinite.

3.1.13 General properties of Groups

Theorem 1 The identity element of a group is a unique element.

Proof: Let us suppose e and e' are two identity elements of a group G, with respect to operation o.

Then e o e' = e if e' is identity.

and e o e' = e if e is identity.

But e o e' = e is unique element of G, therefore,

e o e' = e and e o e' = e $\Rightarrow$ e = e'

Hence the identity element in a group is unique.

Theorem 2 The inverse of each element of a group is unique, i.e., in a group G with operation o' for every a $\in$ G , there is only one element a^{-1} such that

a^{-1} o a = e = a o a^{-1}, e being the identity.

Proof: Let a be any element of a group G and let e be the identity element. Suppose there exists a^{-1} and a' two inverses of a in G then

a^{-1} o a = e = a o a^{-1} and a' o a = e = a o a'

Notes

Now, we have

$$a^{-1}o(a\ o\ a')=a^{-1}o\ e \qquad [\because a\ o\ a'=e]$$

$$=a^{-1} \qquad [\because e \text{ is identity}]$$

$$\text{Also,}(a^{-1}o\ a)o\ a'=e\ o\ a' \qquad [\because a^{-1}\ o\ a=e]$$

$$=a' \qquad [\because e \text{ is identity}]$$

But $a^{-1}o(a\ o\ a')=(a^{-1}o\ a)oa'$ as in a group composition is associative.

$\therefore a^{-1} = a'$

Theorem 3 If the inverse of a is a-1, then the inverse of a^{-1} is a, i.e., $(a^{-1})^{-1} = a$

Proof: If e is an identity element, we have $a^{-1}o\ a = e$.

$$\Rightarrow (a^{-1})^{-1}o(a^{-1}o\ a)=(a^{-1})^{-1}o\ \text{e} \qquad \left[\because a^{-1}\in G \Rightarrow (a^{-1})^{-1}\in G\right]$$

$$\Rightarrow \left[(a^{-1})^{-1}o\ a^{-1}\right]o\ a=(a^{-1})^{-1}$$

$$[\because \text{ Composition in G is associative and e is identity element}]$$

$$\Rightarrow e\ o\ a=(a^{-1})^{-1} \qquad [\because\ a^{-1}\in G]$$

$$\Rightarrow a=(a^{-1})^{-1}$$

$$\Rightarrow (a^{-1})^{-1}=a$$

Theorem 4 The inverse of the product of two elements of a group G is the product of the inverse taken in the reverse order i.e.,

$$(a\ o\ b)^{-1}=b^{-1}o\ a^{-1}\ \ \forall\ \ a,b\in G$$

Proof: Let us suppose a and b are any two elements of G. If and are inverses of a and b respectively, then

Notes

$$a^{-1} o\, a = e = a\, o\, a^{-1} \quad [e \text{ being the identity element}]$$

and $$b^{-1} o\, b = e = b\, o\, b^{-1}$$

Now, $$(aob)o(b^{-1}oa^{-1}) = [(aob)ob^{-1}]oa^{-1} \quad [\text{by associativity}]$$
$$= [a\, o(b\, o\, b^{-1})]o\, a^{-1} \quad [\text{by associativity}]$$
$$= (a\, o\, e)o\, a^{-1} \quad [\because\ b \circ b^{-1} = e]$$
$$= a\, o\, a^{-1} \quad [\because\ a \circ e = a]$$
$$= e \quad [\because\ a \circ a^{-1} = e]$$

Also, $$(b^{-1}o\, a^{-1})o(a\, o\, b) = b^{-1}o[a^{-1}o(a\, o\, b)] \quad [\text{by associativity}]$$
$$= b^{-1}o[(a^{-1}o\, a)o\, b] \quad [\text{by associativity}]$$
$$= b^{-1}o(e\, o\, b) \quad [\because\ a^{-1} \circ a = e]$$
$$= b^{-1}o\, b \quad [\because\ e\, o\, b = b]$$
$$= e \quad [\because\ b^{-1}o\, b = e]$$

Hence, we have

$$(b^{-1}o\, a^{-1})o(a\, o\, b) = e = (a\, o\, b)\circ(b^{-1}o\, a^{-1})$$

Therefore, by definition of inverse, we have

$$(a\, o\, b)^{-1} = b^{-1}o\, a^{-1}$$

Theorem 5 Cancellation laws hold good in a group i.e., if a, b, c, are any element of G, then

$$a\, o\, b = a\, o\, c \Rightarrow b = c \quad [\text{Left cacellation law}]$$

and

$$b\, o\, a = c\, o\, a \Rightarrow b = c \quad [\text{Right cacellation law}]$$

Proof: Let $a \in G$. Then

$a \in G \Rightarrow G$ such that $a \circ a^{-1} = e = a^{-1} \circ a$,

where e is the identity element

Now, let us assume that

Notes

$a \text{ o } b = a \text{ o } c$

then $\quad a \text{ o } b = a \text{ o } c \Rightarrow a^{-1} o (a \text{ o } b) = a^{-1} o (a \text{ o } c)$

$\Rightarrow (a^{-1} o\ a) o\ b = (a^{-1} o\ a) o\ c \qquad$ [by associative law]

$\Rightarrow e\ o\ b = e\ o\ c \qquad [\because a^{-1} o\ a = e]$

$\Rightarrow b = c$

$b \text{ o } a = c \text{ o } a$

$\Rightarrow (b\ o\ a) o\ a^{-1} = (c\ o\ a) o\ a^{-1}$

$\Rightarrow b\ o (a\ o\ a^{-1}) = c\ o (a\ o\ a^{-1}) \qquad$ [by associative law]

$\Rightarrow b\ o\ e = c\ o\ e \qquad [\because a^{-1} o\ a = e]$

$\Rightarrow b = c$

Theorem 6 If G is a group with binary operation o and if a and b are any elements of G, then the linear equations

$a\ o\ x = b$ and $y\ o\ a = b$

have unique solutions in G.

Proof:

$$\because \quad a \in G \Rightarrow a^{-1} \in G,$$

and $\quad a^{-1} \in G, b \in G \Rightarrow a^{-1} o\ b \in G$

Substituting $a^{-1} o\ b$ for x in the equation $a\ o\ x = b$, we obtain

$$a\ o (a\ o\ a^{-1}) = b$$

$$\Rightarrow (a\ o\ a^{-1}) o\ b = b$$

$$\Rightarrow e\ o\ b = b$$

$$\Rightarrow b = b$$

Thus, $x = a^{-1} o\ b$ is a solution of the equation $a\ o\ x = b$.

Let us suppose that the equation $a\ o\ x = b$ has two solutions given by

$$x = x_1 \text{ and } x = x_2$$

Then $a\ o\ x_1 = b$ and $a\ o\ x_2 = b$

$$\Rightarrow a\ o\ x_1 = a\ o\ x_2 = b$$

$$\Rightarrow x_1 = x_2 \qquad \text{[by left cancellation law]}$$

In a similar manner, we can prove that the equation

$y\ o\ a = b$

Notes

Has the unique solution

$$y = b \; o \; a^{-1}$$

Theorem 7 If corresponding to any element $a \in G$; there is an element which satisfies one of the conditions

$$a + 0_a = a \text{ or } 0_a + a = a$$

Then it is necessary that $0_a = a$, where 0 is the identity element of the group.

Proof: Since 0 is the identity element,

We have $a + 0 = a$ ....(i)

given $a + 0_a = a$ ....(ii)

Hence from (i) and (ii)

$$a + 0_a = a + 0$$

$$0_a = a \qquad \text{[by left cancellation law]}$$

Again, we have

$$0 + a = a \qquad ....(iii)$$

given $0_a + a = a$ ....(iv)

Hence from (iii) and (iv)

$$0_a + a = 0 + a$$

$$0_a = a \qquad \text{[by right cancellation law]}$$

Activity:

1. Show that multiplication is a binary operation on the set $A = \{1, -1\}$ but not on $B = \{1, 3\}$.
2. Show that the set of all integers..., –2, –1, 0,1, 2, 3,... is an infinite abelian group with respect to the operation of addition of integers.
3. Show that C, the set of all non-zero complex numbers is a multiplicative group.
4. Prove that the set of cube roots of unity is an abelian finite group with respect to the multiplication.
5. Prove that if every element of a group G is its own inverse, then G is abelian group.

Summary:

- A binary operation 'o' on G is a mapping from G × G to G, i.e., o : G × G → G where the image of (a, b) of G × G under 'o' i.e., o (a, b), is denoted by a o b .
- A binary operation over a set G is said to be commutative, if for every pair of elements a, b ∈ G, a o b = b o a
- A binary operation o on set G is called associative if a o(b o c) = (a o b)oc for all a, b, c ∈ G .

Notes

- Let o and o' be two binary operations defined on a set G. Then operation o' is said to be left distributive with respect to operation o if

  $a\ o'(b\ o\ c)=(a\ o'b)o(a\ o'c)$ for all $a,b,c\in G$

  and is said to be right distributive with respect to o if,

  $(b\ o\ c)\ o'\ a=(b\ o'\ a)o(c\ o'\ a)$ for all $a,b,c\in G$

- A composition o in a set G is said to admit of an identity if there exists an element for all $e\in G$ such that

  $a\ o\ e=a=e\ o\ a\ \ \forall\ a\in G$

- An element $a\in G$ is said to have its inverse with respect to certain operation o if there $b\in G$ such that

  $a\ o\ b=e=b\ o\ a$

- An algebraic structure (G, o) where g is a non-empty set with an operation 'o' defined on it is said to be a group, if the operation satisfies the following axiom (called group axiom)

  (a) Closure axiom (G_1)

  (b) Associative axiom (G_2)

  (c) Identity axiom (G_3)

  (d) Inverse axiom (G_4)

- If an algebraic system (G, o) satisfies only G_1 axiom, it is called a groupoid or a quasi-group.
- If an algebraic system (G, o) satisfies only G_1 and G_2 axiom, it is called a semigroup or a demi-group.
- A semi-group which also satisfies G3 is called a monoid. Monoid is usually denoted by M.
- A group (G, o) is said to be abelian or commutative if the composition 'o' is commutative, i.e.,

  $a\ o\ b = b\ o\ a\ \forall\ a,b \in G$

- A group which is not abelian is called non-abelian.
- If a group has a finite number of distinct elements, it is called the finite group otherwise an infinite group.
- The number of elements or members in a finite group is called the order of the group. The order of an infinite group is always infinite.

Unit - 3.2: Subgroups and other Groups

Recall Session:

In the previous unit, you studied about:

- The binary operations
- Types of binary operations
- Definition of Group, Groupoid, Semi-group, Monoid and Quaternian Group
- Definition of Abelian group, non-abelian-group, finite group, non-finite group and order of a group
- General properties of groups

Unit Outcome:

At the end of this unit, you will learn to

1. Define Subgroup
2. Define Properties of Subgroup
3. Define the order of an element of a group
4. Define Cyclic group and Properties of Cyclic group

3.2.1 Introduction

In the previous unit, we studied about binary operations, types of binary operations, group, semi-group, groupoid, monoid, quaternian group, finite and infinite group, order of a group, abelian and non-abelian group and general properties of groups also. In this unit, we introduce the idea of obtaining smaller groups from a given group, and in doing so, we sometimes utilise the services of a single element of the group. These smaller groups are called subgroups of the given group, and many of them are of great importance because they act as true representative of the parent group in the sense that they retain the characteristics properties of the group. In this unit, we will also discuss about order of an element of group and cyclic group.

3.2.2 Subgroup

A non-empty subset H of a group G is said to be a subgroup of G if the composition in G induces a composition in G induces a composition in H and if H is a group for the induced composition.

The sub groups (i) consisting of the identity element alone, and (ii) the group G itself are always present in a group g.

These are, however, trivial subgroups. A sub-group other than these two is known as proper sub-group.

A complex is any subset of a group, whether it is a sub-group or not.

It is easy to prove that

Notes

a. The identity of a sub-group is the same as that of the group.

b. The inverse of any element of a sub-group is the same as the inverse of the element regarded as a member of the group.

c. The order of any element of a sub-group is the same as that of the element regarded as a member of the group.

Example 3.2.1 The additive group of integers is a sub-group of the additive group of rational numbers.

Example 3.2.2 The multiplicative group of positive rational numbers is a sub-group of the multiplicative group of non-zero real numbers.

Necessary and sufficient condition

The necessary and sufficient conditions for a subset of a group to be a sub group are stated in the following two theorems.

Theorem 1 A subset H of a group G is a sub-group iff

i. $(a \in H, b \in H) \Rightarrow a \text{ o } b \in H$

ii. $a \in H \Rightarrow a^{-1} \in H$

Proof: Suppose H is a group Of G then H must be closed with respect to composition o in G, i.e., $a \in H, b \in H \Rightarrow a \text{ o } b \in H$.

Let $a \in H$ and a^{-1} be the inverse of a in G. As H itself is a group, each element of H will posses inverse in it, i.e., $a \in H, a^{-1} \in H$.

Thus, the condition is necessary. Now let us examine the sufficiency of the condition.

a. **Closure axiom:** Since $a \in H, b \in H \Rightarrow a \text{ o } b \in H$. Hence closure axiom is satisfied with respect to the operation o.

b. **Associativity:** Since the elements of H are also the elements of G, the composition is associative in H also.

c. Existence of identity: The identity of the subgroup is similar to the identity of the group because,

given $\quad a \in H, a^{-1} \in H$

Hence $\quad a \text{ o } a^{-1} \in H$

i.e., $\quad e \in H$

d. Existence of inverse: Since $a \in H \Rightarrow a^{-1} \in H, \forall a \in H$.

Therefore, each element of H possesses inverse.

Thus, H itself is a group for the composition in G. Hence H is a sub-group.

Theorem 2 A necessary and sufficient condition for a non-empty subset H of a group g to sub group is that $a \in H, b \in H \Rightarrow a.b^{-1} \in H$ where b^{-1} is the inverse of b in G.

Proof: The condition is necessary Suppose H is a sub-group of G and let $a \in H, b \in H$.

Now each element of H must posses inverse because H itself is a group.

$b \in H \Rightarrow b^{-1} \in H$

Also, H is closed under the composition o in G. Therefore,

$a \in H, b \in H \Rightarrow a.b^{-1} \in H$

The condition is sufficient. As $a \in H, b \in H \Rightarrow a.b^{-1} \in H$, we have to prove that H is a subgroup.

a. Closure axiom: Let a,b $\in$ H then $b \in H \Rightarrow b^{-1} \in H$

Therefore, by the given condition

$a \in H, b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H$

$\Rightarrow a \text{ o } b \in H$

Thus, H is closed with respect to the composition o in G.

b. **Associative axiom:** Since the elements of H are also the elements of G the composition is associative in H.

c. **Existence of identity:** Since

$a \in H, a^{-1} \in H \Rightarrow aa^{-1} \in H$

$\Rightarrow e \in H$

Thus, the identity element belongs to H.

d. **Existence of Inverse:** Let $a \in H$, then

$e \in H, a \in H \Rightarrow ea^{-1} \in H$

$\Rightarrow a^{-1} \in H$

Thus, each element of H possesses an inverse.

Hence H itself is a group for the composition o in a group G.

3.2.2.1 Properties of subgroups

Theorem 1 The intersection of two sub-groups of a group G is a subgroup of G.

Proof: Let H_1 and H_2 be any two subgroups of G.

Then, $H_1 \cap H_2 \neq \phi$ (because at least the identity element e is common in both H_1 and H_2.).

Now, to prove that $H_1 \cap H_2$ is a subgroup of G,

it is sufficient to show that

$a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow a.b^{-1} \in H_1 \cap H_2$, (because o being composition in G)

Since $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

and $b \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$

H_1 and H_2 are sub-groups of G, we see that

$a \in H_1, b \in H_1 \Rightarrow a.b^{-1} \in H_1$ and similarly

$a \in H_2, b \in H_2 \Rightarrow a.b^{-1} \in H_2$

Thus, $a.b^{-1} \in H_1, a.b^{-1} \in H_2, a.b^{-1} \in H_1 \cap H_2$

Notes

Hence, $a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow a.b^{-1} \in H_1 \cap H_2$

which establishes that $H_1 \cap H_2$ is a sub-group of G.

Theorem 2 The union of two sub groups is not necessarily a sub group.

Proof: Let G be the additive group of integers, and let

$H_1 = \{0, \pm 2, \pm 4, \pm 6.....\}$

$H_2 = \{0, \pm 3, \pm 6, \pm 9.....\}$

Then, H_1 and H_2 are subgroups of G, but

$H_1 \cup H_2 = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9.....\}$

which is not a group. It is concluded that the closure property is not satisfied. For, 2 + 3 = 5, which does not belong to $H_1 \cup H_2$.

The set $H_1 \cap H_2 = \{0, \pm 6, \pm 12.....\}$, which is certainly a group.

Hence, the union of two subgroups is not necessarily a group.

Theorem 3 The union of two subgroups is a sub-group if and only if one is contained in the other.

Proof: Let H_1 and H_2 be two subgroups of a group G.

i. Let $H_1 \subset H_2$ or $H_2 \subset H_1$

Then $H_1 \cup H_2 = H_2$ or H_1

But H_1, H_2 are subgroups, so that $H_1 \cup H_2$ is also a sub-group.

ii. Next suppose $H_1 \cup H_2$ is a sub-group.

To prove that $H_1 \subset H_2$ or $H_2 \subset H_1$, assume if possible that

$H_1 \not\subset H_2$ or $H_2 \not\subset H_1$

Now,

$H_1 \not\subset H_2 \Rightarrow \exists s \in H_1$ and $s \notin H_2$ ....(i)

$H_2 \subset H_1 \Rightarrow \exists t \in H_1$ and $t \notin H_1$ ....(ii)

From (i) and (ii), it follows that

$s \in H_1 \cup H_2, t \in H_1 \cup H_2$

Since $H_1 \cup H_2$ is a sub-group, we see that

St = k (say) is also an element of $H_1 \cup H_2$

But

$st = k \in H_1 \cup H_2$

$\Rightarrow st = k \in H_1 \cup H_2$

Suppose, $st = k \in H_1$

Then $t = s^{-1}k \in H_1$ [H_1 is a sub-group, $s^{-1} \in H_1$]

This contradicts (ii).

Hence either $H_1 \not\subset H_2$ or $H_2 \not\subset H_1$.

Theorem 4 A finite subset H of a group G is a subgroup of G, if a, b $\in$ H $\Rightarrow$ ab $\in$ H.

Proof: Let H be a subgroup of G.

H is a subgroup $\Rightarrow$ H is a group

$\Rightarrow$ H is closed for the given operation,

i.e., a, b $\in$ H $\Rightarrow$ ab $\in$ H, $\forall$a, b $\in$ H.

The condition is necessary. Let H be a non-empty finite subgroup of G such that,

a, b $\in$ H $\Rightarrow$ ab $\in$ H

To show that H is a subgroup of G, we are to show that H is a group.

(G_1) **Closure axiom:** If a, b $\in$ H, then

$\Rightarrow$ ab $\in$ H [by the given condition]

(G_2) **Associative axiom:** If a, b, c $\in$ H then for all

a, b $\in$ H $\Rightarrow$ a, b, c $\in$ G, as H $\in$ G

(ab)c = a(bc) [bu associative law in G]

a, b $\in$ H $\Rightarrow$ a, b, c $\in$ G, as H $\in$ G

$\therefore$ In H, (ab)c = a(bc), $\forall$a, b, c $\in$ H

(G_3) **Identity axiom:** Let e be the identity of G.

Then, a $\in$ H. Hence, a $\in$ H $\Rightarrow a^2 = a.a \in$ H

$a^3 = a^2.a \in$ H

$\Rightarrow a^4 = a^3.a \in$ H

Hence if a $\in$ H, then a, a^2, a^3,..... a^r,.... a^s,.... $\in$ H

But H is a finite set. Hence, out of these elements many will be the same elements, because if they are distinct, H will not be a finite set.

Hence for some positive integers r and s (r > s)

$a^r = a^s \Rightarrow a^r.a^{-s}$

$\Rightarrow a^{r-s} = a^0 = e$, where e is the identity of G

$\Rightarrow a^{r-s} = e$, where r–s is the positive integer

$\Rightarrow e = a^{r-s} \in H$

$\Rightarrow e \in H$

Hence, identity axiom is satisfied.

(G_4) **Inverse axiom:** If r – s $\geq$ 1, then r – s $\geq$ 0

Notes

Notes

$\Rightarrow a^{r-s-1} \in H$, where e is the identity of G

$\Rightarrow a^{r-s} a^{-1} \in H$, where r–s is the positive integer

$\Rightarrow ea^{-1} \in H$

$\Rightarrow a^{-1} \in H$

Hence, $a \in H \Rightarrow a^{-1} \in H \ \forall a \in H$

i.e., every element of H is inversible.

So, H is a subset Of G.

3.2.3 Order of An Element of a Group

If G is a group and $a \in G$, the order (or period) of a is the least positive integer n such that

$a^n = e$

If there exists no such integer, we say that a is of infinite order or of zero order.

We shall use the notation o(a) for the order of a.

Example 3.2.3 Find the order of each element of the multiplicative group G where .

Solution 1 is the identity element, its order is 1.

Now, $(-1)^1 = -1$, $(-1)^2 = (-1)(-1) = 1$

Hence the order of –1 is 2.

Again $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

Therefore, the order of i is 4.

Similarly,

$(-i)^1 = -i$, $(-i)^2 = -1$,

$(-i^3) = i$, $(-i^4) = 1$

Hence order of –i is 4.

Example 3.2.4 If the elements A, B and AB of a finite group of order 2, prove that AB = BA.

Solution Given $A^2 = e$, $B^2 = e$ and $(AB)^2 = e$, where e is the identity element.

Now,

$(AB)^2 = e \Rightarrow (AB)(AB) = e$

$\Rightarrow A(AB)(AB) = Ae$

$\Rightarrow A^2 (BA)B = A$

$\Rightarrow \{e(BA)B\}B = AB$

$\Rightarrow (BA)B^2 = AB$

Notes

$\Rightarrow$ (BA)e = AB

$\Rightarrow$ (BA) = AB

Thus AB = BA

3.2.4 Cyclic Group

A group G is called cyclic if, for some $a \in G$, every element $x \in G$ is of the form a^n, where n is some integer. Thus, the element a is called a generator of G.

Example 3.2.5 The multiplicative group $\{1,\omega, \omega^2\}$ is cyclic. The generators are ,ω and ω^2.

Example 3.2.6 The multiplicative group of n^{th} roots of unity is cyclic, a generator being $e^{\frac{2\pi i}{n}}$.

3.2.4.1 Properties of Cyclic Groups

Theorem 1 Every cyclic group is abelian.

Proof: Let a be a generator of a cyclic group G and let $a^r, a^s \in G$ for any $r, s \in I$ then

$a^r. a^s = a^{r+s} = a^{s+r}$ [$r + s = s + r$ for $r, s \in I$]

$= a^s.a^r$

Thus, the operation is commutative, and hence the cyclic group G is abelian.

Theorem 2 The order of a cyclic group is same as the order of its generator.

Proof: Let the order of a generator of a cyclic group be n, then

$a^{ns} = e$ while $a^s \neq e$ for $0 < s < n$

When $s > n$, $s = nq + r$, $0 \leq r \leq n$ (say), we observe that

$a^s = a^{nq+r} = (a^n)^q.a^r = e^q.a^r$

$= e.a^r = a^r$

Thus, there are exactly n elements in the group by a^r, where $0 \leq r \leq n$.

Therefore, there are n and only n distinct elements in the cyclic group, i.e., the order of the group is n.

Theorem 3 if a is generator of a cyclic group G, then a-1 is also a generator of G.

Proof: Using the multiplicative notation for the operation in G, we have

$$G=(a)=\{a^k \mid k\in \mathrm{I}\}$$

Let a^r be an arbitrary element of G, where $r \in \mathrm{I}$. Now, we can write $a^r = \left(a^{-1}\right)^{-r}$

Where –r is certainly an integer, because $r \in \mathrm{I}$. Therefore, every integral power of a is some integral power of a-1 and vice-versa. We, thus, see that every element of G can also be generated by a-1, and so a-1 is also a generator of G.

Notes

Example 3.2.7 Find the generators of the cyclic group

(a, a^2, a^3, a^4, a = e}

Solution Here order of the group is 5. Now,

$$(1,5)=1,\ (2,5)=1,\ (3,5)=1,\ (4,5)=1$$

So that 1, 2, 3, 4 are relatively prime to 5. Hence, the generators of the given cyclic group are a, a^2, a^3, a^4.

Activity:

Prove the following:

- Show that a non-empty subset H of a group H will be subgroup iff $HH^{-1} = H$.
- Prove that the order of every element of a finite group is finite.
- Prove that the order of an element of a group is the same as that of its inverse a^{-1}.
- Prove that if the element a of a group G is of order n, then $a^m = e$ iff n is divisor of m.
- Prove that the order of a cyclic group is same as the order of its generator.
- Prove that if G is a finite cyclic group and a is a generator of G, then G has elements. Also, if o(a) = n, then

$$G=(a)=\{e,a,a^2,....,a^{n-1}\}$$

Summary:

- A non-empty subset H of a group G is said to be a subgroup of G if the composition in G induces a composition in G induces a composition in H and if H is a group for the induced composition.
- If G is a group and a∈G, the order (or period) of a is the least positive integer n such that $a^n = e$.
- A group G is called cyclic if, for some a∈G, every element x∈G is of the form a^n, where n is some integer. Thus, the element a is called a generator of G.

Further Reading:

1. Dr. P.K. Mittal “Abstract Algebra”, S.J. Publications, Chhipi Tank, Meerut.

Exercise:

Check your progress

1. A binary operation, denoted by ‘o’, on a non-empty set G is the mapping

(a) $o: G \to G$

(b) $o: G\times G \to G\times G$

(c) $o: G\times G \to G$

(d) $o: G \to G\times G$

Notes

2. Which of the following is a group?
   (a) (Z, +)
   (b) (Z, -)
   (c) (Z, ×)
   (d) (Z, ÷)
3. A finite set G with a binary composition, which is associative, and is a group, if and only if
   (a) Commutative law holds
   (b) Cancellation law holds
   (c) Reversal law holds
   (d) None of these
4. If $G = R - \{-1\}$ and 0 is defined by aob = a + b + ab, ∀ a, b, ∈ G then the identity element for the group (G, 0) is
   (a) –1
   (b) 1
   (c) ±1
   (d) 0
5. The inverse of an element a in the group considered in Q4 above is
   (a) $\frac{1}{a}$
   (b) -a
   (c) $\frac{a+1}{a}$
   (d) $-\frac{a}{(a+1)}$
6. The set of residue classes modulo 5, under the addition of residue classes modulo 5, is
   a) an infinite abelian group
   (b) an infinite non-abelian group
   (c) a finite abelian group
   (d) a finite non-abelian group
7. Set of all odd integer, with respect to addition, forms a
   (a) group
   (b) abelian group

Notes

(c) not a group

(d) none of these

8. For the set of all positive rational numbers under the composition defined $a\ o\ b = \frac{ab}{2}$, the identity element is

(a) 1

(b) –1

(c) 2

(d) –2

9. In Q8, the inverse of the element a is

(a) $-a$

(b) $\frac{1}{a}$

(c) $\frac{2}{a}$

(d) $\frac{4}{a}$

10. The element of the group, which is the inverse of itself, is

(a) Identity element

(b) Inverse element

(c) Every element

(d) None of these

11. If a, b, c are in G, then ab = ac $\Rightarrow$ b = c is the consequence of

(a) Reversal law

(b) Left cancellation law

(c) Right cancellation law

(d) None of these

12. Set $G = \{1, \omega, \omega^2\}$ where ω is an imaginary cube root of unity, with respect to the multiplication is

(a) Simple group

(b) Abelian group

(c) Not a group

(d) None of these

13. The multiplication group $G = \{1, i, -1, -i\}$ is

(a) Non-abelian

Notes

(b) Non-cyclic

(c) Imaginary

(d) Cyclic

14. The order of the group S_3 (of all permutation on three symbols) with respect to the product of permutations is

(a) 3

(b) 9

(c) 6

(d) 1

15. The group S_3 in Q 14 above is

(a) Finite and abelian

(b) Infinite and abelian

(c) Finite and non-abelian

(d) Infinite and non-abelian

16. The order of every element of a finite group is

(a) 0

(b) 1

(c) Finite

(d) Infinite

17. If an element a of a group is of order n, and p is prime to n, then the order of a^p is

(a) p^n

(b) n^p

(c) np

(d) n

18. The multiplication of a group of residues classes 1,3,5,7 (mod 8) is

(a) Finite and cyclic

(b) Finite but not cyclic

(c) Abelian and cyclic

(d) Non- abelian and cyclic

19. How many generators does a cyclic group of order n have?

(a) As many integers lie between 0 and n

(b) As many even integers lie between 0 and n

(c) As many odd integers lie between 0 and n

(d) As many prime numbers lie between 0 and n

Notes

20. Any group G of order 3 is
   (a) Abelian
   (b) Non-abelian
   (c) Cyclic
   (d) Non-cyclic
21. In any infinite cyclic group, the number of generators is
   (a) Infinite
   (b) 0
   (c) 1
   (d) 2
22. Element of the group $G=\{a,a^2,a^3,a^4,a^5,a^6=e\}$ which can be used as generators of the group
   (a) a
   (b) a and a5
   (c) a, a3 and a5
   (d) a2, a4 and a6
23. Every cyclic group is
   (a) Cyclic
   (b) on-cyclic
   (c) Abelian
   (d) Non-abelian
24. In the multiplication group G, $G=\{a,a^2,a^3,a^4,a^5,a^6=e\}$ the order of a^4 is
   (a) 1
   (b) 2
   (c) 3
   (d) 4
25. If the elements a, b of a group commute and $o(a)=m,\ o(b)=n$, where $(m,n)=1$ then $o(ab)$ is
   (a) $m+n$
   (b) $m-n$
   (c) mn
   (d) $\frac{m}{n}$

Notes

26. In an additive group of integers, order of every element except 0 is
    (a) Finite
    (b) Infinite
    (c) 0
    (d) 1
27. Of element of the group {0, 1, 2, 3, 4, 5} the composition being addition modulo 6, the order of 2 is
    (a) 1
    (b) 2
    (c) 3
    (d) 4
28. The center Z of a group G is
    (a) Sub-group
    (b) Normal-subgroup
    (c) Ring
    (d) Field
29. Every sub-group of an abelian group is
    (a) Group
    (b) Abelian group
    (c) Sub group
    (d) Normal-subgroup
30. Every quotient group of a cyclic group is
    (a) Cyclic group
    (b) Normal-subgroup
    (c) Quotient group
    (d) None of these
31. Every quotient group is
    (a) Cyclic group
    (b) Quotient group
    (c) Abelian group
    (d) Normal-subgroup

Notes

Answer Keys (Exercise)

Question	Answer	Question	Answer	Question	Answer
1	c	2	a	3	b
4	d	5	d	6	c
7	c	8	c	9	d
10	a	11	b	12	b
13	d	14	c	15	c
16	c	17	d	18	b
19	d	20	c	21	d
22	b	23	a	24	c
25	c	26	a	27	c
28	b	29	d	30	a
31	c				

Module – 4: Graph Theory

Notes

Course Contents:

- Graph
- Multi-graph
- Complete graph
- Bi graph
- Degree
- Isomorphic graph
- Euler graph
- Hamiltonian graph
- Bipartite graph

Key Learning Objectives:

At the end of this block, you will be able to:

1. Define Graph
2. Define Multi-graph
3. Define Complete graph
4. Define Bi-graph
5. Define Degree of Vertex
6. Define Isomorphic graph
7. Define Euler graph
8. Define Hamiltonian graph
9. Define Bipartite graph

Structure:

UNIT 4.1: Graph Theory

4.1.1 Introduction

4.1.2 Definition of Graph

4.1.3 Simple Graph

4.1.4. Multi-graph

4.1.5 Complete graph

4.1.6 Bi-graph

4.1.7 Degree of Vertex

Notes

4.1.8 Isomorphic graph

4.1.9 Euler graph

4.1.9.1 Euler Path

4.1.10 Hamiltonian graph

4.1.11 Bipartite graph

4.1.12 Complete Bipartite graph

Unit - 4.1: Graph Theory

Notes

Unit Outcome:

At the end of this unit, you will learn

- Define Graph, Simple graph, Multi-graph, Complete graph and Bi graph
- Define degree of Vertex
- Define Isomorphic graph, Euler graph, Hamiltonian graph and Bipartite graph

4.1.1 Introduction

Graph theory is an applied branch of mathematics. Due to simplicity, it has a wide range of applications in operations research, genetics, physical, biological and social sciences, engineering, computer science and in many other areas. The large portion of Graph theory have been motivated by recreational mathematics and the study of games. In the present world, we consider graph a mathematical model, solve the appropriate graph-theoretic problem, and then interpret the solution in terms of the original problem. In this unit, we will discuss about graph, multi-graph, complete graph, Bi graph and degree.

4.1.2 Definition of graph

A graph is a pictorial representation consisting of points called vertices and lines called edges; each edge joins exactly two vertices.

A graph $G=(V,E)$ consists of a finite set denoted by V or V(G) and a collection E or E(G) of unordered pairs (u, v) of distinct elements from V. Each element of V is called a vertex or node or point and each element of E is called an edge or a line or a link.

The cardinality of V, i.e., the number of vertices, is called the order of graph G and denoted by |V|. The cardinality of E, i.e., the number of edges, is called the size of the graph and denoted by |E| .

Let $V(G)=\{v_1,v_2,....,v_n\}$ and $E(G)=\{e_1,e_2,....,e_m\}$ and, be the set of vertices and edges of a graph G. Each edge $e_k=E(G)$ is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i and v_j are called the end vertices of ek. Fig 4.1.1 represents a graph with six vertices and ten edges.

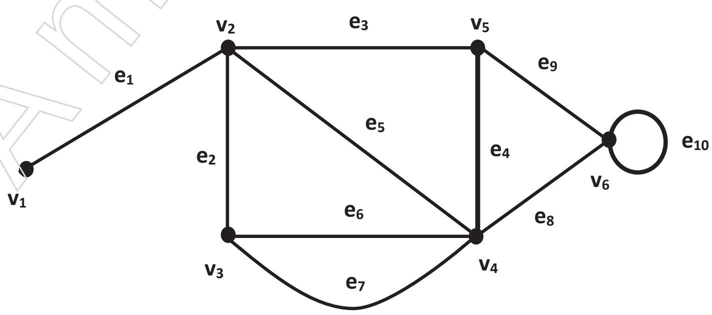


Fig. 4.1.1 Graph

Notes

4.1.3 Simple Graph

A graph without self-loops and parallel edges is called a simple graph.

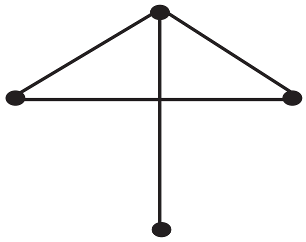


Fig. 4.1.2 Simple Graph

4.1.4 Multi-graph

A multi-graph M consists of a finite non-empty set V of vertices and a set E of edges, where two vertices of M are joined by a finite number of edges (possibly zero). If two or more edges join the same pair of vertices, then these edges are called parallel edges. In a multigraph, an edge is also permitted to join a vertex to itself. Such an edge is called a loop. If a loop v joins a vertex v to itself then, i.e. is said to be a loop at v. There can be any finite number of loops at the same vertex in a multigraph.

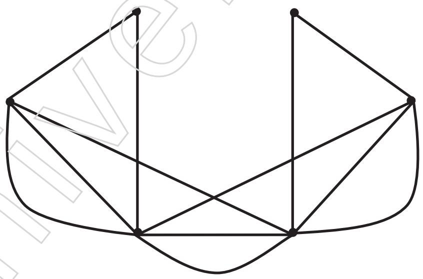


Figure 4.1.3Multigraph

4.1.5 Complete graph

If in a simple graph there exist an edge between each and every pair of vertices, then the graph is said to be a complete or full graph. A complete graph of n vertices is represented by K_n.

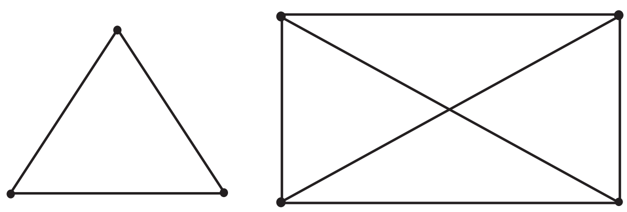


Figure 4.1.4 Completes graphs of 3 and 4 vertices

4.1.6 Bi-graph

A bi-graph consists of two orthogonal structure: a place graph that describes the nesting of entities, e.g. a phone inside a room, and link graph that provides non-local hyperlinks between entities, e.g. allowing phone entities to communicate regardless of location.

Notes

Bi-graphs are compositional structures, i.e we can build larger bi-graphs. Composition of bi-graphs consists of placing regions in sites, and connecting inner and outer-faces on like-names.

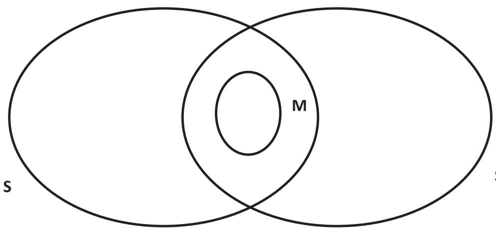


Fig. 4.1.5 Bi-graph

4.1.7 Degree of Vertex

The degree of any vertex v of G is the number of edges incident with vertex v. Each self-loop is counted twice. Degree of a vertex is always a positive number and is denoted by deg (v). The minimum and maximum degree of vertices in V(G) are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

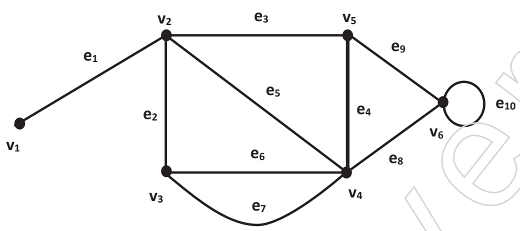


Fig. 4.1.6 Graph with 6 vertices to edges

4.1.8 Isomorphic Graph

Two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ and are said to be isomorphic if there exists a function $f:V_1 \rightarrow V_2$ such that

f is one -to one onto i.e., f is bijective.

{a, b} is an edge in e_1, if and only if {f(a), f(b)} is an edge in e_2 for any two elements a, b, $\in V_1$

The condition (b) says that if vertices a and b are adjacent in G_1 the f(a) and f(b) are adjacent in G_2. In other words, the function f preserves adjacency and consequently the corresponding vertices

In G_1 and G_2 will also have the same degree. Any function f with the above properties is called an isomorphism between G_1 and G_2.

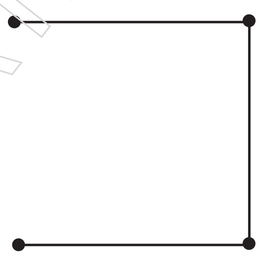


Fig. 4.1.7 Graph G_1

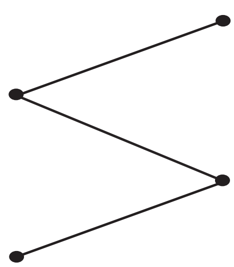


Figure 4.1.8 Graph G_2

The graph of above pair is an example of isomorphic graph.

Notes

4.1.9 Euler Graph

In a graph G, if a closed walk contains all the edges of the graph G, then the walk is called an Euler line or Euler cycle.

A graph G is called as Euler graph if it contains an Euler line in G.

In an Euler line each edge appears exactly once and the Euler graph is always connected.

4.1.9.1 Euler Path

An open walk in a graph G is called an Euler Path if it contains all the edges of graph G.

Theorem 1. A given connected graph G contains an Euler line (i.e., graph G is Euler graph) if all the vertices of G are of even degree.

Proof: Suppose G is a connected graph with n vertices (v_1, v_2,...v_n) and m-edges (e_1, e_2,...e_n) also suppose that G is an Euler graph. Therefore, G must contain an Euler line (which is a closed walk). Let's assume that $v_1 e_1\ v_2 e_2\ v_3 e_3\ e_4\ \ldots\ e_i\ v_i\ \ldots\ e_{m-1}\ v_m\ e_m\ v_1$ is the Euler line in the given graph.

It can be seen easily that every time the line traces a new vertex v it goes through two new edges incident on v. So, every time we trace a vertex, it's degree increases by 2. Hence, if G is an Euler graph, the degree of every vertex is even.

Conversely, let us assume that all the vertices of G are of even degree. To prove that G is an Euler graph, we need to construct an Euler line in G. Start with an arbitrary vertex v and tracing the edges of G such that no edges appear more than once. Since G contains all the vertices of even degree, we can exit from the same vertex we enter. This means that the walk cannot stop at any vertex but only v.

Now if walk w, say starting and ending at v, traced all the edges of the graph G, walk is an Euler line and the graph is an Euler graph. But if walk w does not contain all edges of G, then remove all the edges which are in walk w from the graph G, we are left with a graph which is a subgraph of G. The degree of the vertex in v in even because graph G and walk w have all their vertices of even degree.

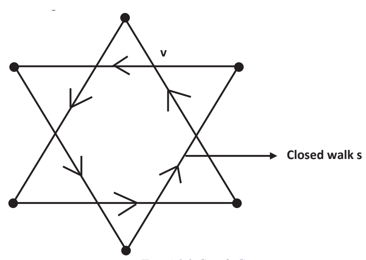


Fig. 4.1.9 Graph G

Notes

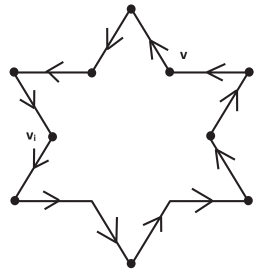
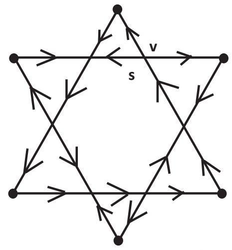


Fig. 4.1.10 $G' = G - w$



Fia. 4.1.11 Euler graph

Theorem 2. A connected graph G is an Euler graph if it can be de decomposed into edge-disjoint circuits.

Proof: Let G be a connected graph and can be decomposed into edge-disjoint circuits $c_1, c_2, ..., c_n$.

Then $G = c_1 \cup c_2 \cup ..... \cup c_n$

i.e., G is the union of edge-disjoint circuits. We know that the degree of every vertex in a circuit is two. Hence the degree of every vertex in G is a multiple of 2 (i.e., even). Therefore, G is an Euler graph (Theorem 1).

Conversely, let us suppose that G is an Euler graph. Now we need to show that it can be decomposed into edge-disjoint circuits. Consider an arbitrary vertex v_1 in G. Since the degree of v_1 is even, there must be at least two edges incident on v_1. Let us say once the edge is between v_1 and v_2 .

Similarly, there are at least two edges incident on v_2, and one of the edge is between v_2 and v_3. In this way we tracing a circuit starting and ending at v_1. Now remove the circuit from the graph G, we are left with a subgraph of G (need not necessarily connected), whose vertices are of even degree. Continue this process until no edge is left. Hence prove the theorem.

4.1.10 Hamiltonian Graph

Hamiltonian graphs are named after Sir William Hamilton, an Irish mathematician who introduced the problems of finding a circuit in which all vertices of a graph appear exactly once.

A circuit in a graph G that contains each vertex in G exactly once except for the starting and ending vertex that appears twice is known as the Hamiltonian cycle.

A graph G is called a Hamiltonian cycle if it contains a Hamiltonian cycle.

A Hamiltonian path is a simple path and contains all vertices of g where the endpoints may be distinct.

Notes

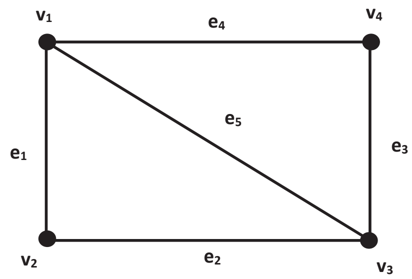
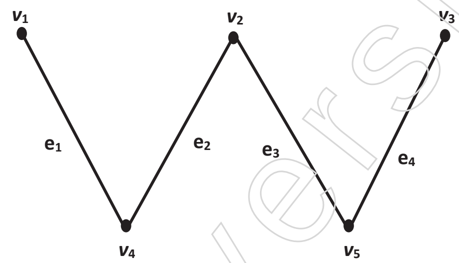


Fig. 4.1.12 Hamiltonian Cycle or Graph

4.1.11 Bipartite Graph

A graph G = (V, E) is called a Bipartite graph if its's vertex set V(G) can be partitioned into two non-empty disjoint subsets $V_1(G)$ and $V_2(G)$ in such a way that each edge e $\in$ E(G) has it's one endpoint in $V_1(G)$ and other endpoint in $V_2(G)$. The partition $V = V_1 \cup V_2$ is called a bipartition of G.



$$V_1(G) = \{v_1, v_2, v_3\}$$
$$V_2(G) = \{v_4, v_5\}$$
$$V_1 \cap V_2 = \phi$$

Fig. 4.1.13 Bipartite Graph

4.1.12 Complete Bipartite graph

If each vertex of $V_1(G)$ is joined with each vertex $V_2(G)$, then the graph G is called complete bipartite graph and is denoted by Km, n where m and n are the numbers of vertices in $V_1(G)$ and $V_2(G)$ respectively.

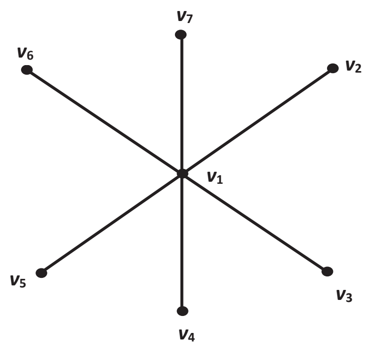


Fig. 4.1.14 $K_{1,6}$ or Star graph

$K_{1,n}$ is called a Star graph.

Miscellaneous Exercise:

Notes

Ex-4.1 A graph G has 21 edges, 3 vertices of degree 4 and other vertices are of degree 3. Find the number of vertices in G.

Solution: Let there be n vertices in G. Out of these n vertices 3 are of degree 4 and (n-3) vertices are degree 3.

So, Degree of graph G= Sum of degree of all vertices

deg = 3 × 4 + 3 × (n – 3) ......(1)

But we also know that

deg = 2 × No. of edges

2 × 21 = 42 ......(2)

Hence from equation (1) and (2)

$$12+3(n-3)=42$$
$$n-3=\frac{42-12}{3}$$
$$n=13$$

So, the total number of vertices in G are 13.

Ex-4.2 A graph G has 8 edges. Find the number of vertices, if the degree of each vertex is 2.

Solution: Let there be n vertices in graph G.

So, Degree of graph G = Sum of degree of all vertices

deg = 2 × n ....(1)

But, we know that

deg = 2 × No. of edges

= 2 × 8 = 16 ...(2)

Hence, from equation (1) and (2)

$$2\times n=16$$
$$n=\frac{16}{2}$$
$$=8$$

So, the total number of vertices in G are 8.

Ex-4.3 The maximum number of edges in a bipartite graph on 12 vertices is ________?

Solution: We know,

Notes

Maximum possible number of edges in a bipartite graph on ‘n’ vertices $=\frac{1}{4}\times n^2$

Substituting n = 12, we get

Maximum possible number of edges in a bipartite graph on 12 vertices

$$=\frac{1}{4}\times 12^2$$
$$=\frac{144}{4}$$
$$=36$$

Therefore, Maximum possible number of edges in a bipartite graph on 12 vertices = 36

Ex-4.4 A simple graph G has 24 edges and degree of each vertex is 4. Find the number of vertices.

Solution: Given

Number of edges = 24

Degree of each vertex = 4

Let number of vertices in the graph = n

We know that

Degree of graph G = Sum of degree of all vertices = 2 × No. of edges

Substituting the value and we get

$$n\times 4=2\times 24$$
$$n=2\times 6$$
$$n=12$$

Thus, number of vertices in the graph = 12

Ex-4.5 A graph has 24 edges and degree of each vertex is k, then which of the following is possible number of vertices?

(a) 20

(b) 15

(c) 10

(d) 8

Solution: Given

Number of edges = 24

Degree of each vertex = k

Let number of vertices in the graph = n

We know that

Degree of graph G = Sum of degree of all vertices = 2 × No. of edges

Notes

Substituting the value and we get

$$n \times k = 2 \times 24$$

$$k = \frac{48}{n}$$

We know that the degree of any vertex must be a whole number. So, here only 8 is possible value of 'n' which gives the whole value of 'k'.

Hence option (d) is correct.

Ex-4.6 Which of the following graphs has an Eulerian circuit?

(a) Any k-regular graph where k is an even number.

(b) A complete graph on 90 vertices

(c) The complement of a cycle on 25 vertices

(d) None of these

Solution: We know that , a graph has Eulerian circuit if following conditions are true.

i. All vertices with non-zero degree are connected. We don't care about vertices with zero degree because they don't belong to Eulerian Cycle or Path.

ii. All vertices have even degree.

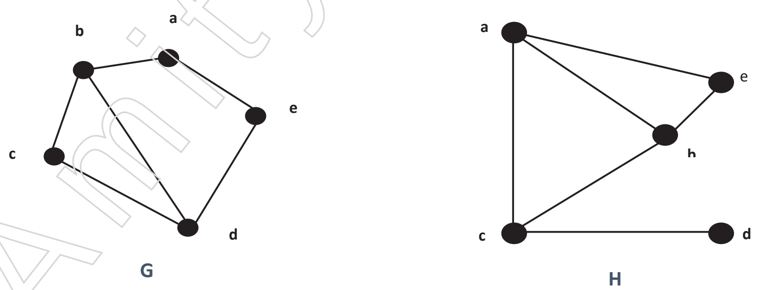
Any k-regular graph where k is an even number, is not Eulerian as a k regular graph may not be connected.Hence property (ii) is true , (i) may not.

A complete graph on 90 vertices is not Eulerian because all vertices have degree as 89.Hence property (ii) is false.

The complement of a cycle on 25 vertices is Eulerian. In a cycle of 25 vertices, all vertices have degree as 2. In complement graph, all vertices would have degree as 22 and graph would be connected.

Hence option (c) is a correct answer.

Ex-4.7 Determine whether the graphs G and H are isomorphic.

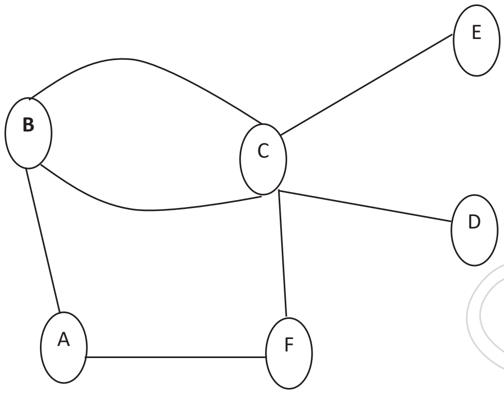


Solution: No, they are not isomorphic, because they differ in the degrees of their vertices.

Vertex d in right graph is of degree one, but there is no such vertex in the left graph.

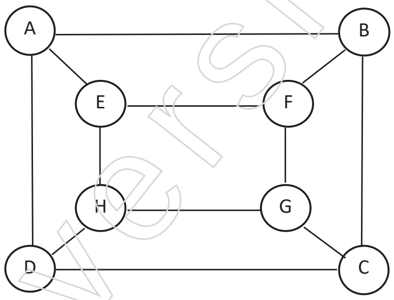
Notes

Ex-4.8 Does the following graph have a Hamiltonain Circuit?



Solution: No the above graphs does not have a Hamiltonian circuit as there are two vertices with degree one in the graph.

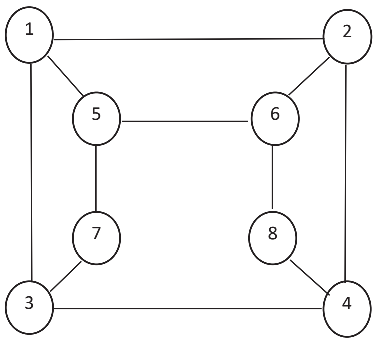
Ex-4.9 Does the following graph is a Hamiltonain graph?



Solution: The graph contains both a Hamiltonian path (ABCDHGFE) and a Hamiltonian circuit (ABCDHGFEA).

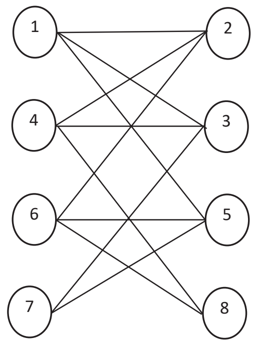
Since graph contains a Hamiltonian circuit, therefore it is a Hamiltonian Graph.

Ex– 4.10 Is the following graph a bipartite graph?



Solution: The graph may be redrawn as

Notes



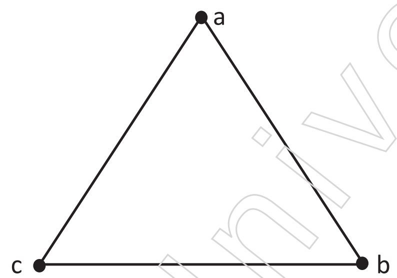
This graph consists of two sets of vertices. The two sets are
X = {1, 4, 6, 7} and Y = (2, 3,5, 8).

The vertices of set X are joined only with the vertices of set Y and vice-versa. Also, any two vertices with in the same set are not joined. This satisfies the definition of a bipartite graph.

Therefore, given graph is a bipartite graph.

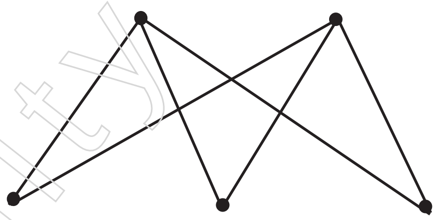
Ex-4.10 Draw a figure of complete graph K3.

Solution:

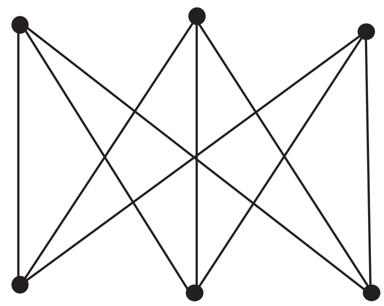


Ex-4.11 Draw a figure of complete bipartite graph $K_{2,3}$ and $K_{3,3}$.

Solution: The graph of $K_{2,3}$



The graph of $K_{3,3}$.



Notes

Ex-4.12 Show that the maximum number of edges in a simple graph with n vertices is n(n – 1)/2.

Solution: We know that the maximum degree of any vertex in a simple graph of n vertices is (n – 1).

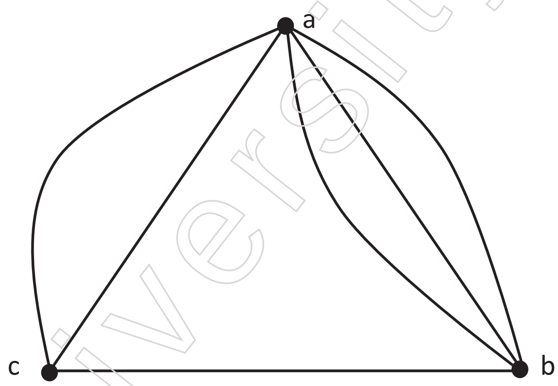
So, the total maximum degree of the graph of n vertices = n(n – 1)

By the theorem, the sum of degree of all vertices in G is twice the number of edges in G.

Hence the maximum number of edges $e = \frac{n(n-1)}{2}$

Ex-4.13 Draw a figure of Simple Multigraph.

Solution:



Summary:

- A graph is a pictorial representation consisting of points called vertices and lines called edges; each edge joins exactly two vertices.
- A graph without self-loops and parallel edges is called a simple graph.
- A multi graph M consists of a finite non- empty set V of vertices and a set E of edges, where two vertices of M are joined by a finite number of edges (possibly zero).
- If in a simple graph there exist an edge between each and every pair of vertices, then the graph is said to be a complete or full graph.
- A bi-graph consists of two orthogonal structure: a place graph that describes the nesting of entities, e.g. a phone inside a room, and link graph that provides non-local hyperlinks between entities, e.g. allowing phone entities to communicate regardless of location.
- The degree of any vertex v of G is the number of edges incident with vertex v. Each self-loop is counted twice.
- Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a function $f: V_1 \to V_2$ such that
- f is one -to one onto i.e., f is bijective.

Notes

- $\{a, b\}$ is an edge in E_1, if and only if $\{f(a), f(b)\}$ is an edge in E_2 for any two elements $a, b \in V_1$.
- A graph G is called as Euler graph if it contains an Euler line in G.
- An open walk in a graph G is called an Euler Path if it contains all the edges of graph G.
- A graph G is called a Hamiltonian cycle if it contains a Hamiltonian cycle.
- A graph $G=(V,E)$ is called a Bipartite graph if its's vertex set V(G) can be partitioned into two non-empty disjoint subsets $V_1(G)$ and $V_2(G)$ in such a way that each edge $e \in E(G)$ has it's one end point in $V_1(G)$ and other end point in $V_2(G)$.
- If each vertex of $V_1(G)$ is joined with each vertex $V_2(G)$, then the graph G is called complete bipartite graph and is denoted by Km, n where m and n are the number of vertices in $V_1(G)$ and $V_2(G)$ respectively.

Further Reading:

1. Kalika Patrai, "Graph Theory", S.K. Kataria & Sons, New Delhi

Exercise:

Check your progress

1. Which of the following statements for a simple graph is correct?
   (a) Every path is a trail
   (b) Every trail is a path
   (c) Every trail is a path as well as every path is a trail
   (d) Path and trail have no relation
2. Which of the following properties does a simple graph not hold?
   (a) Must be connected
   (b) Must be unweighted
   (c) Must have no loops or multiple edges
   (d) Must have no multiple edges
3. What is the maximum number of edges in a bipartite graph having 10 vertices?
   (a) 24
   (b) 21
   (c) 25
   (d) 16
4. Which of the following is true?
   (a) A graph may contain no edge and many vertices
   (b) A graph may contain no edges and no vertices
   (c) A graph may contain no edges and no vertices
   (d) A graph may contain no vertices and many edges

Notes

5. For a given graph G having v vertices and e edges which is connected and has no cycles, which of the following statements is true?
   (a) v = e
   (b) v = e + 1
   (c) v + 1 = e
   (d) v = e – 1
6. For which of the following combinations of the degree of vertices would the connected graph be eulerian?
   (a) 1, 2, 3
   (b) 2, 3, 4
   (c) 2, 4, 5
   (d) 1, 3, 5
7. Which of the following ways can be used to represent a graph?
   (a) Adjacency List and Adjacency Matrix
   (b) Incidence Matrix
   (c) Adjacency list, Adjacency Matrix as well as Incidence Matrix
   (d) No way to represent
8. A graph is collection of
   (a) Rows and Columns
   (b) Vertices and edges
   (c) Equations
   (d) None of these
9. The degree of any vertex of graph is
   (a) The number of edges incident with vertex
   (b) Number of vertex in a graph
   (c) Number of vertices adjacent to that vertex
   (d) Number of edges in a graph
10. If the origin and terminus of a walk are same, the walk is known as
   (a) Open
   (b) Closed
   (c) Path
   (d) None of these
11. In a graph if   means
   (a) u is adjacent to v is not adjacent to u
   (b) e begins at u and ends at v
   (c) u is predecessor and v is successor

(d) both b and c

12. A graph with n vertices will definitely have a parallel edge or self loop if the total number of edges are

(a) greater than n – 1

(b) less than n (n - 1)

(c) greater than n (n – 1)/2

(d) less than n (n – 1)/2

13. A vertex of a graph is called even or odd depending upon

(a) Total number of edges in a graph is even or odd

(b) Total number of vertices in a graph is even or odd

(c) Its degree is even or odd

(d) None of these

14. The maximum degree of any vertex in a simple graph with n vertices is

(a) n - 1

(b) n + 1

(c) 2n – 1

(d) N

15. How many onto (or surjective) functions are there from an n-element ($n \geq 2$) set to a two-element set?

(a) 2n

(b) 2n – 1

(c) 2n – 2

(d) 2(2n – 2)

16. Circle has

(a) No vertices

(b) Only a vertex

(c) 8 vertices

(d) None of these

17. The complete graph with four vertices has k edges where k is

(a) 3

(b) 4

(c) 5

(d) 6

18. Choose the most appropriate definition of plane graph

(a) A graph drawn in a plane in such a way that any pair of edges meet only at their end vertices

(b) A graph drawn in a plane in such a way that if the vertex set of graphs set of

Notes

Notes

graphs can be partitioned into two non-empty disjoint subset x and y in such a way that each edge of g has one end in X and one end in Y.

(c) A simple graph which is isomorphic to Hamiltonian graph

(d) None of these

19. Length of the walk of a graph

(a) The number of vertices in walk w

(b) The number of edges in walk w

(c) Total number of edges in a graph

(d) Total number of vertices in a graph

20. Which type of graph has all the vertex of the first set connected to all the vertex of the second set?

(a) Bipartite

(b) Complete Bipartite

(c) Cartesian

(d) Pie

Answer Keys (Exercise)

Question	Answer	Question	Answer	Question	Answer
1	a	2	a	3	c
4	b	5	b	6	a
7	c	8	b	9	a
10	b	11	d	12	a
13	c	14	a	15	c
16	a	17	d	18	a
19	b	20	b		

Module – 5: Data Analysis

Notes

Course Contents:

- Data and Statistical Data
- Frequency Distribution
- Graphical Representation
- Measure of Central Tendency
- Measure of Dispersion
- Kurtosis
- Skewness

Key Learning Objectives:

At the end of this block, you will be able to:

1. Define Data and Statistical Data
2. Classify Frequency Distribution
3. Construct various types of Graph
4. Describe Measure of the Central tendency
5. Describe Measure of Dispersion
6. Define Skewness
7. Define Kurtosis

Structure:

UNIT 5.1: Data and their Representation

5.1.1 Introduction

5.1.2 Data and Statistical data

5.1.3 Variable

5.1.4. Arrangement of Raw data

5.1.5 Frequency Distribution

5.1.6 Graphical Representation of Data

5.1.7 Types of Graph

Unit-5.2: Measure of the Central Tendency

5.2.1 Introduction

5.2.2 Arithmetic Mean or Average

5.2.3 Properties of Arithmetic Mean

5.2.4 Geometric Mean

Notes

Unit - 5.1: Data and Their Representation

Notes

Unit Outcome:

At the end of this unit, you will learn

- Define data and statistical data
- Classify the Frequency Distribution
- Construct various types of Graph

5.1.1 Introduction

Statistics is concerned with the scientific method for collecting, organising, summarising, presenting and analysing data (datum) as well as with drawing valid conclusions and making reasonable decisions on the basis of such analysis.

In this unit, we will discuss data, statistical data, the arrangement of raw data frequency distribution and graphical representation of the data.

5.1.2 Data and Statistical Data

Data is a collection of information, but it is in raw form. When data is processed, it becomes information. In short, meaningful, logical, and processed data is called information. Data is numerical, character, symbols, or any other kind of information. Data is plural and datum is the singular form.

The term ‘statistics’ is used in plural sense (i.e., as statistical data) and singular sense (i.e., as statistical method).

Statistics in the plural sense refers to numerical data of any phenomenon placed in relation to each other.

For example, numerical data relating to population, production, price level, national income, crimes, literacy, unemployment etc.

According to prof. Horace Secrist “By statistics we mean the aggregate of all facts affected to a marked extent by multiplicity of causes numerically expressed, enumerated or estimated according to a reasonable standard of accuracy, collected in a symmetric manner for a predetermined purpose and placed in relation to each other”.

5.1.3 Variable

A quantity which can vary or change from one individual to another is called a variable. The values which a variable takes are called observations on the variable or simply observations or variate values.

The name ‘variable’ actually comes from a number symbol X, which may represent some characteristics and can be replaced by a number from real number.

There are two kinds of variables:

1. **Discrete Variables:** A variable which takes only finite or denumerable many distinct

Notes

values is known as discrete variable. For example, the number of student in your class, the number of males in your family, etc.

2. **Continuous Variable:** A variable which can theoretically assume all values within a certain interval or intervals is called a continuous variable. For example, height, temperature, weight, etc.

5.1.4 Arrangement of Raw data

The data that are given in its original form often called ungrouped data. It is very difficult for the mind to grasp the significance of the raw data. The process of arranging the data in a logical or systematic order is called seriation and data so arranged is called a statistical series or simply series.

If the data in arranged in ascending or descending order of magnitude it is said to be arranged in array.

For example, Let the height (in cm) of students in your class be 150, 151, 149, 175, 180, 164, 151, 145, 170, 160.

The height of these students in ascending order are 145, 149, 150, 151, 151, 160, 164, 170, 175, 180.

The number of times each value of a variable occurs is known as its frequency. Here two students have same height. So, the frequency of 151 is 2.

5.1.5 Frequency Distribution:

Frequency distribution is an arrangement of data according to the number (called frequency) possessing the individual or grouped values the variable.

or

A tabular form of the data in which the frequencies of the values of a variable are given along with them is called a frequency distribution.

1. **Univariate Frequency Distribution:** A frequency distribution which shows the frequency of occurrence of different values of a single variable is called a univariate frequency distribution.
2. **Bivariate Frequency Distribution:** A frequency distribution based on two variables is known as bivariate frequency distribution.
3. **Discrete Frequency Distribution:** A frequency distribution which is formed by distinct values of a discrete values of a discrete variable or a continuous variable is called a discrete frequency distribution.
4. **Grouped Frequency Distribution:** A frequency distribution which is obtained by dividing the entire range of given observations on a discrete or continuous variable into groups and distributions the frequencies over these groups is called a grouped frequency distribution.

The groups are called classes and the boundary ends are called class limits. For class 10-20, say 10 is the lower limit and 20 is the upper limit. The difference between the upper and lower limits of a class or class interval is called its magnitude or width of the class interval. The number of observations are falling within a particular class is called its frequency or class frequency. The value of the variable which lies mid-way between the upper and lower limits is called mid- point or mid value that class.

Example 5.1.1 Given below are the marks obtained by 24 students in an examination:

18, 17, 16, 24, 25, 19, 41, 22, 32, 42, 44, 21, 43, 26, 28, 40, 29, 30, 37, 27,49, 27, 34, 31

Make a frequency table for the above distribution.

Solution: The frequency table for the given data is is:

Marks	Frequency
10-20	4
20-30	9
30-40	5
40-50	6
	Total =24

Cumulative Frequency and cumulative frequency table

The cumulative frequency of a class interval is the sum of frequencies of all classes up to that class.

Thus, cumulative frequency table for example 1 is as follows:

Marks	Frequency	Cumulative Frequency
10-20	4	4
20-30	9	4 + 9 = 13
30-40	5	4 + 9 +5 = 18
40-50	6	4 + 9 +5 +6 = 24
	N = 24	

This table can also be written in the form of ‘less than’ cumulative frequency table as:

Marks	Cumulative Frequency
Less than 10	0
Less than 20	4
Less than 30	13
Less than 40	18
Less than 50	24

5.1.6 Graphical Representation of a Frequency Distribution

It is often useful to represent frequency distribution by means of a graph which makes the unwieldy data intelligible and coveys to the eye the general run of the

Notes

observations. The graphs have a more lasting effect on the brain. When data of two items are compared with one another it is always easier to compare through graphs.

5.1.7 Types of Graphs

Generally, the following types of graphs are used in representing frequency distributions:

1. Histogram
2. Frequency Polygon
3. Frequency Curve
4. Cumulative Frequency Curve or the Ogive

1. **Histogram**

In this graph, the rectangles are drawn with class-intervals as bases and their heights are proportional to the frequencies of respective classes.

Example 5.1.2 A histogram of the following given data is shown below

Daily wages (in rupees)	Frequency
0-100	2
100-200	4
200-300	10
300-400	4
400-500	3
500-600	8
600-700	1
700-800	5
800-900	11
900-1000	2

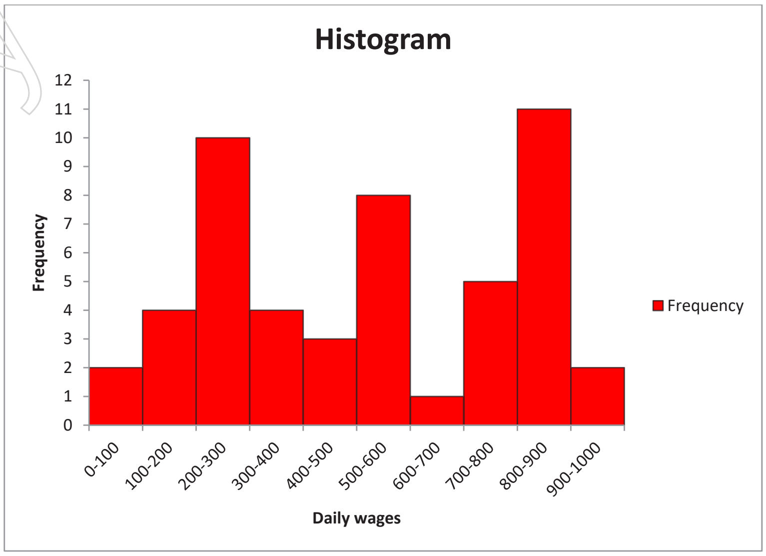


Fig. 5.1.1 Histogram

2. Frequency Polygon

If the various points are obtained by plotting the central values of the class-intervals as x-coordinates and the respective frequencies as the y-coordinates, and these points are joined by straight lines and they form a polygon called frequency polygon.

In a frequency polygon the individuals or variables of each class are assumed to be concentrated at the mid-point of the class-interval.

Example 5.1.3 A frequency polygon for the given following data

Class	Frequency
0-10	2
10-20	4
20-30	10
30-40	4
40-50	3
50-60	8
60-70	1
70-80	5
80-90	11
90-100	2

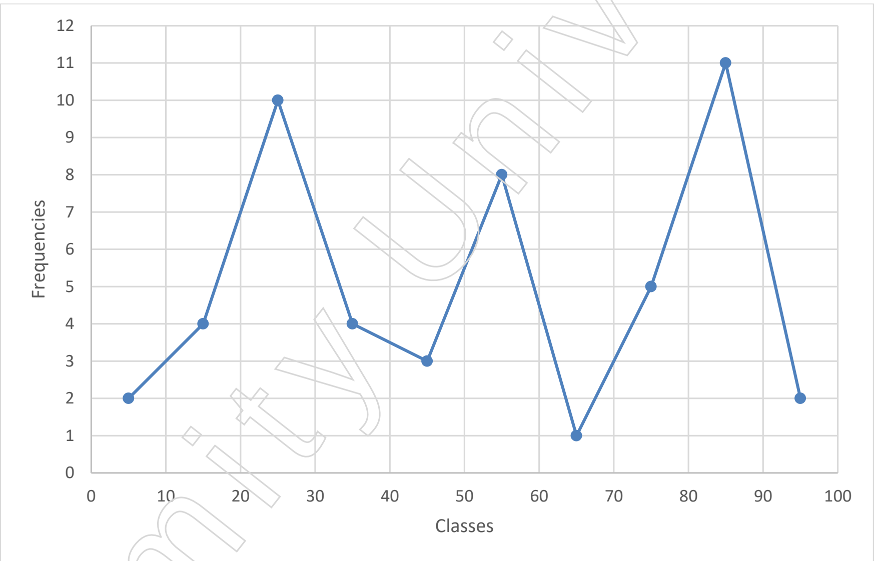


Fig. 5.1.2 Frequency polygon

3. Frequency Curve

A frequency curve is drawn by smoothing the frequency polygon. It is smoothed in such a way that the sharp turns are avoided. A frequency polygon, if smoothed further so as to minimise sudden changes, results into a continuous smooth curve known as frequency or smooth frequency curve. The curve should begin and end at the base line.

Example 5.1.4 A frequency curve for the given following data

Notes

Class	Frequency
0-10	2
10-20	4
20-30	10
30-40	4
40-50	3
50-60	8
60-70	1
70-80	5
80-90	11
90-100	2

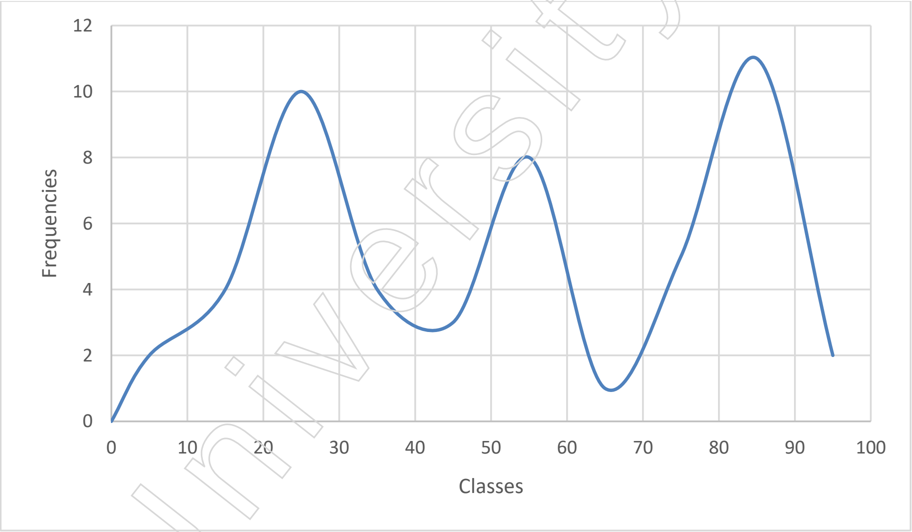


Fig. 5.1.3 Frequency curve

4. Cumulative Frequency Curve or the Ogive

If from a cumulative frequency table, the upper limits of the class taken as x-coordinates and the cumulative frequencies as the y-coordinates and the points are plotted, then these points when joined by straight lines, we obtain less than type cumulative frequency polygon.

If more than cumulative frequency is plotted against the corresponding lower limits of each class and the points plotted are joined by straight lines, we obtain more than type cumulative frequency polygon.

However, when the points plotted are joined by a free hand smooth curve, we obtain cumulative frequency curve.

The cumulative frequency polygon is often called an ogive.

Example 5.1.5 A cumulative frequency polygon for the given following data

Class	Frequency	Cumulative frequency
0-10	2	2
10-20	4	6
20-30	10	16
30-40	4	20
40-50	3	23
50-60	8	31
60-70	1	32
70-80	5	37
80-90	11	48
90-100	2	50

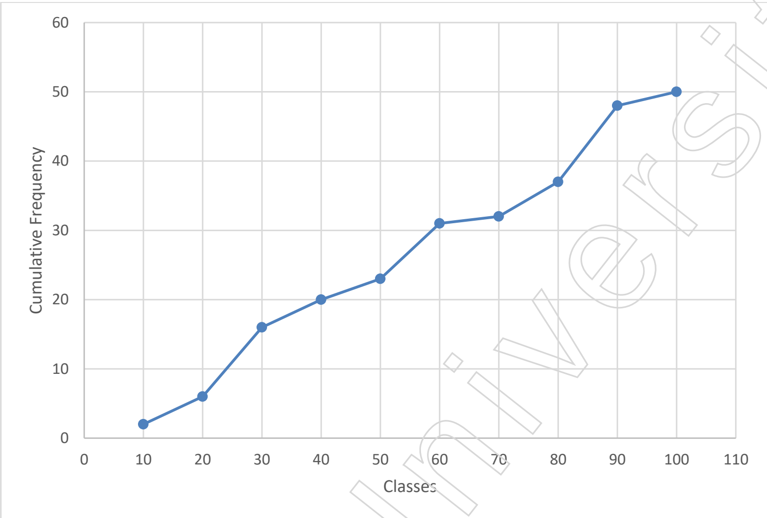


Fig. 5.1.4 Cumulative Frequency

Activity:

1. Construct an ogive curve for the following frequency distribution of Cotton Mils in Bombay according to the quantities of cotton consumed-

Cotton consumed in thousand candles	No. of Mils
0-2	5
2-4	13
4-6	12
6-8	11
8-10	8
10-12	4
12-14	1
14-16	3
16-18	1

Notes

18-20	1
over 20	2

1. The following table shows the frequency distribution for the number of students per teacher in 750 colleges and professional schools-

Students x	Frequency F
1	7
4	46
7	165
10	195
13	189
16	89
19	28
22	19
2	9
528	3

Summary:

- Data is collection of information, but it is in raw form. When data is processed it becomes information.
- A quantity which can vary or change from one individual to another is called a variable.
- A variable which takes only finite or denumerable many distinct values is known as discrete variable.
- A variable which can theoretically assume all values within a certain interval or intervals is called a continuous variable.
- A tabular form of the data in which the frequencies of the values of a variable are given along with them is called a frequency distribution.
- A frequency distribution which shows the frequency of occurrence of different values of a single variable is called a univariate frequency distribution.
- A frequency distribution based on two variables is known as bivariate frequency distribution.
- A frequency distribution which is formed by distinct values of a discrete values of a discrete variable or a continuous variable is called a discrete frequency distribution.
- A frequency distribution which is obtained by dividing the entire range of given observations on a discrete or continuous variable into groups and distributions the frequencies over these groups is called a grouped frequency distribution.

Notes

Unit - 5.2: Measure of the Central Tendency

Recall Session:

In the previous unit, you studied about:

- The Data and Statistical Data
- The Variable and Arrangement of Raw Data
- The Frequency Distribution
- The Graphical Representation of Data
- The types of Graph

Unit Outcome:

At the end of this unit, you will learn to

1. Define Arithmetic Mean or Average
2. Describe Properties of Arithmetic Mean
3. Define Geometric Mean
4. Define Median, Positional Measure and Mode
5. Describe the relationship between Mean, Mode and Median

5.2.1 Introduction

In the previous unit, we studied about data and statistical data, variable, arrangement of raw data, frequency distribution, graphical representation of data and the types of graph. When two or more different series of the same type are compared, it is not enough to classify and to tabulate the observations. To make the data more comprehensive it is often desirable

to define quantitatively the characteristics of frequency distribution.

There are four fundamental characteristics in which similar frequency distribution may differ. One of the characteristic is central tendency. In this unit, we will discuss about arithmetic mean or average, properties of arithmetic mean, median, positional measure, mode and the relationship between mean, mode and median.

5.2.2 Arithmetic Mean or Average

Arithmetic mean of a group of observations is the quotient obtained by dividing the sum of all the observations by their number. Thus, arithmetic mean denoted by is

$$\bar{x} = \frac{\text{Sum of all the observations}}{\text{Number of Observations}}$$

Formula:

Individual Series: If X_1, X_2, X_3,...., X_n are the n values of a variate x, then the arithmetic mean (A.M.) is given by

Notes

$$A.M. = \frac{x_1 + x_2 + ... + x_n}{n}$$

$$= \frac{\sum x}{n}$$

Discrete Frequency Distribution: if the value x1 occurs f1 times, the value x2 occurs f2 times, and so on, then

$$A.M. = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n}$$

$$= \frac{\sum_{i=1}^{n} f_ix_i}{N}$$

Where N = $f_1 + f_2 + ... + f_n$ = Total frequency.

Methods of calculating Arithmetic mean

There are three methods of calculating arithmetic mean

a. Direct method
b. Short-Cut method
c. Step-deviation method

These methods are applicable to any type of series.

a. **Direct method:**

$$A.M. = \frac{\sum x}{n}$$

b. **Short -cut Method:** In this method, first we assumed any number, say A, (often called assumed mean). Then

$$\bar{x} = A + \frac{\sum f(x - A)}{\sum f}$$

$$= A + \frac{\sum f(x - A)}{N}$$

c. **By Step-Deviation Method**

$$\bar{x} = A + h\frac{\sum_{i=1}^{n} f_iu_i}{\sum_{i=1}^{n} f_i}$$

Where,

h = Class size

$$u_i = \frac{x_i - A}{h}$$

A = Assumed mean

$\sum_{i=1}^{n} f_i$ = Sum of the frequencies given, can be denoted by N.

5.2.3 Properties of arithmetic mean

1. The algebraic sum of the deviations of all the variate values from their mean is zero.
2. If every value of the variable is increased by same constant a, then arithmetic mean is also increased by a.
3. Arithmetic mean is not independent of the change of origin and scale.
4. The sum of the squares of the deviations of all the values taken about their mean is minimum.

Example 5.2.1 Find the arithmetic mean of first n natural numbers.

Solution: Let the sum of all-natural numbers are denoted by $\sum x$.

$$\sum x = 1+2+3+.....+n$$

$$= \frac{n(n+1)}{2}$$

Hence Mean,

$$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{n(n+1)}{2n}$$

$$= \frac{n+1}{2}$$

Example 5.2.2 Compute arithmetic mean of the following by both direct and short-cut methods-

Class	20-30	30-40	40-50	50-60	60-70
Frequency	8	26	30	20	16

Solution:

Class	Mid-value X	F	fx	d = x - A	Fd
20-30	25	8	200	-20	-160
30-40	35	26	910	-10	-260
40-50	45	30	1350	0	0
50-60	55	20	1100	10	200

Notes

60-70	65	16	1040	20	320
		$\sum f = 100$	$\sum fx = 4600$		$\sum fd = 100$

i. By direct method

$$\bar{x} = \frac{\sum fx}{N}$$
$$= \frac{4600}{100}$$
$$= 46$$

ii. By short-cut method,

Here, assumed mean = 45

$$\bar{x} = A + \frac{\sum fd}{N}$$
$$= 45 + \frac{100}{100}$$
$$= 46$$

5.2.4 Geometric Mean

The n^{th} root of product of the values is called geometric mean.

i. Geometric Mean for Ungrouped Data

If $X_1, X_2, X_3, \ldots, X_n$ be n observations, then geometric mean is given by

$$G = (x_1.x_2.......x_n)^{\frac{1}{n}}$$

$$\log G = \frac{1}{n}(\log x_1 + \log x_2 + ... + \log x_n)$$
$$= \frac{1}{n}\sum_{i=1}^{n} \log x_i$$

$$G = \text{antilog}\left[\frac{1}{n}\sum_{i=1}^{n} \log x_i\right]$$

ii. Geometric Mean for Grouped Data

If $X_1, X_2, X_3, \ldots, X_n$ be n observations whose corresponding frequencies are f^1, f_2, $f_3, \ldots, f_n$ then geometric mean is given by

$$G = \left(x_1^{f_1}.x_2^{f_2}.......x_n^{f_n}\right)^{\frac{1}{N}}$$

$$G = \text{antilog}\left[\frac{1}{N}\sum_{i=1}^{n} f_i \log x_i\right]$$

Example 5.2.3 Find the geometric mean of the numbers 3, 3^2, 3^3,...., 3^n.

Notes

Solution

$$GM = \left(3.3^2.3^3......3^n\right)^{\frac{1}{n}}$$

$$= 3^{\frac{1+2+.....+n}{n}}$$

$$= 3^{\frac{n(n+1)}{2n}}$$

$$= 3^{\frac{n+1}{2}}$$

5.2.5 Median

When the observations are arranged in ascending or descending order of magnitude, then the middle value is called the median of these observations.

Median is that value of the variable which divides a given series into two parts so that one-half or more of the items are equal to or less than it.

Formula to compute Median of the given data

1. Formula for individual series: Let $X_1, X_2, X_3, ..., X_n$ be the n values of a variable written in ascending order of magnitude. Then median denoted by M_e or M or Md is given by

Median = Value of the middle item

a. When n is odd, then

$$\text{Median} = \frac{n+1}{2}\text{th term}$$

b. When n is even, then

$$\text{Median} = \frac{1}{2}\left\{\frac{n}{2}\text{th term} + \left(\frac{n}{2}+1\right)\text{th term}\right\}$$

2. Formula for grouped data:

$$\text{Median} = l + \frac{\frac{N}{2} - C.F.}{f} \times h$$

where,

l = lower limit of the median class

f = the frequency of the median class

h=width of the median class

C.F.= cumulative frequency

N = total frequency

$\frac{N}{2}$ = median class

5.2.6 Positional Measure

These are the values of the variable which divides total frequency into number of

Notes

equal parts e.g., Median divides the total frequency into two equal parts.

Some of these are as follows

i. Quartiles

Quartile are those values of the variate which divide the total frequency into four equal parts.

Median is that value of the variable which divides the total frequency into two equal parts. When the lower half before the median is divided into two equal parts the value of the dividing variate is called Lower Quartile and is denoted by Q_1.

The value of the variate dividing the upper half is called the Upper Quartile and is denoted by Q_3.

The formula for quartile

$$Q_1 = l + \frac{\frac{N}{4} - C.F.}{f} \times h$$

And

$$Q_3 = l + \frac{\frac{3}{4}N - C.F.}{f} \times h$$

ii. Deciles

Those values of variables which divides the total distribution into ten equal part, are known as deciles.

iii. Percentiles

Those values of variables which divides the total distribution into hundred parts, are known as percentiles.

Example 5.2.4 Calculate median, lower quartile and upper quartile for the following data-

Class	0-4	4-6	6-8	8-12	12-18	18-20
Frequency	4	6	8	12	7	2

Solution

Class	Frequency f	Cumulative Frequency C.F.
0-4	4	4
4-6	6	10
6-8	8	18
8-12	12	30
12-18	7	37
18-20	2	39
Total	39	

Notes

(i) For Median, $\frac{N}{2}=\frac{39}{2}=19.5$

19.5th item lies in the class 8-12. Thus

$$\text{Median}=l+\frac{\frac{N}{2}-C.F.}{f}\times h$$

$$=8+\frac{19.5-18}{12}\times 4$$

$$=8+\frac{1.5\times 4}{12}$$

$$=8+0.5$$

$$=8.5$$

(ii) For lower quartile,

$$\frac{N}{4}=\frac{39}{4}=9.75$$

9.75th item lies in the class 4-6. Thus

$$Q_1=l+\frac{\frac{N}{4}-C.F.}{f}\times h$$

$$=4+\frac{9.75-4}{6}\times 2$$

$$=4+\frac{5.75\times 2}{6}$$

$$=4+1.93$$

$$=5.93$$

(iii) For upper quartile,

$$\frac{3N}{4}=\frac{117}{4}=29.25$$

29.25th item lies in the class 8-12. Thus

Notes

$$Q_3 = l + \frac{\frac{3N}{4} - C.F.}{f} \times h$$

$$= 8 + \frac{29.25 - 18}{12} \times 4$$

$$= 8 + \frac{11.25 \times 4}{12}$$

$$= 8 + 3.75$$

$$= 11.75$$

5.2.7 Mode

The mode or modal value of the given distribution is that value of the variate for which frequency is maximum. It can be denoted by symbol 'M_o'.

Formula of Mode for grouped data

$$M_o = l + \frac{f - f_{-1}}{2f - f_{-1} - f_1} \times h$$

where,

f is the frequency of the modal class

l is the lower limit of the modal class

f_{-1}, f_1 are the frequencies of the classes preceding and following the modal class

h is the class interval

Example 5.2.5 Compute the mode of the following data

Mid value	15	20	25	30	35	40	45	50	55
Frequency	2	22	19	14	3	4	6	1	1

Solution:

Thus, the given data transforms to the continuous series, as given below-

Mid value	Class	Frequency
15	12.5-17.5	2
20	17.5-22.5	22
25	22.5-27.5	19
30	27.5-32.5	14
35	32.5-37.5	3
40	37.5-42.5	4
45	42.5-47.5	6
50	47.5-52.5	1
55	52.5-57.5	1

Here the maximum frequency f = 22 lies in the class 17.5-22.5.

So, 17.5-22.5 is the modal class.

$\therefore l = 17.5, f = 22, f_{-1} = 2, f_1 = 19, h = 5$

Hence, Mode,

$$M_o = l + \frac{f - f_{-1}}{2f - f_{-1} - f_1} \times h$$

$$= 17.5 + \frac{22 - 2}{2 \times 22 - 2 - 19} \times 5$$

$$= 17.5 + \frac{20 \times 5}{23}$$

$$= 17.5 + 4.35$$

$$= 21.85$$

5.2.8 Relationship between Mean, mode and median

Mean > Median > Mode

Mean – Mode = 3(Mean – Median)

Activity:

1. Calculate the arithmetic mean of the distribution-

Variate	5	10	15	20	25	30	35	40	45	50
Frequency	20	43	75	67	72	45	39	9	8	6

2. How is the arithmetic mean affected if every value of the variable is
   a. decreased by same constant a
   b. multiplied by same constant k
   c. increased by same constant b
   d. divided by the same constant h
3. Find the median and quartiles for the following frequency distribution-

Class	Frequency
0-6	5
6-12	11
12-18	25
18-22	20
22-24	15
24-30	18
30-36	12
36-42	6

4. The height of 70 students of a class are given in the following table. Find their mode-

Notes

Height in cms.	Frequency
120-124	2
125-129	5
130-134	8
135-139	15
140-144	20
145-149	10
150-154	5

5. Find the geometric mean for the following frequency distribution-

Class	Frequency
0-10	1
10-20	2
20-30	6
30-40	6
40-50	5

Summary:

- Arithmetic mean of a group of observations is the quotient obtained by dividing the sum of all the observations by their number.
- **Direct method:**

$$A.M. = \frac{\sum x}{n}$$

- Short -cut Method

$$\bar{x} = A + \frac{\sum f(x-A)}{N}$$

- Step-Deviation Method

$$\bar{x} = A + h\frac{\sum_{i=1}^{n} f_i u_i}{\sum_{i=1}^{n} f_i}$$

- The algebraic sum of the deviations of all the variate values from their mean is zero.
- If every value of the variable is increased by same constant a, then arithmetic mean is also increased by a.
- Arithmetic mean is not independent of the change of origin and scale.

- The sum of the squares of the deviations of all the values taken about their mean is minimum.
- The nth root of product of the values is called geometric mean.
- When the observations are arranged in ascending or descending order of magnitude, then the middle value is called the median of these observations.
- Quartile are those values of the variate which divide the total frequency into four equal parts.
- Those values of variables which divides the total distribution into ten equal part, are known as deciles.
- Those values of variables which divides the total distribution into hundred parts, are known as percentiles.
- The mode or modal value of the given distribution is that value of the variate for which frequency is maximum. It can be denoted by symbol 'Mo'.
- Relationship between Mean, mode and median

$$\text{Mean} > \text{Median} > \text{Mode}$$

$$\text{Mean} - \text{Mode} = 3(\text{Mean} - \text{Median})$$

Notes

Unit - 5.3: Measure of Dispersion

Recall Session:

In the previous unit, you studied about:

- Arithmetic Mean or Average
- Properties of Arithmetic Mean
- Geometric Mean
- Median, Positional Measure and Mode
- The relationship between Mean, Mode and Median

Unit Outcome:

At the end of this unit, you will learn

1. Define Measure of dispersion
2. Define Variance
3. Describe the properties of standard deviation
4. Define the Coefficient of Variation

5.3.1 Introduction

In the previous unit we studied about arithmetic mean or average, properties of arithmetic mean, geometric mean, median, positional measure and mode, the relationship between mean, mode and median. The word dispersion is used in two senses in statistics one is the scatteredness of the values of a variable due to variation among themselves is called dispersion and second is the deviations from a measure of central tendency or any other fixed value are not uniform in their size. The scatteredness of these deviations is also referred to as dispersion. So, In this unit we will discuss about measure of dispersion, variance, properties of standard deviation and the coefficient of variation.

5.3.2 Measure of dispersion

The following are the measures of dispersion which are in common use-

1. Range
2. Quartile deviation or semi-interquartile range
3. Mean deviation
4. Standard Deviation

Range

The simplest possible measure of dispersion is the range which is the difference between the greatest and the least values of the variable.

If L and S are largest and smallest observations respectively, then

Notes

Range = L – S

5. **Quartile deviation or semi-interquartile range**

The difference between the upper and lower quartiles i.e., $Q_3 - Q_1$ is known as the interquartile range. Half of the interquartile range is called the quartile deviation or semi-interquartile range and it is denoted by QD.

$$QD = \frac{1}{2}(Q_3 - Q_1)$$

6. **Mean deviation**

Mean deviation of a distribution is the arithmetic mean of the absolute deviation of the terms of the distribution from its statistical mean (A.M., median or mode).

i. **For Ungrouped or individual Series**

If X_1, X_2, X_3,...., X_n are n observations, then mean deviation from average, A (usually mean, mode or median) is

$$MD = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

$$= \frac{\sum_{i=1}^{n} |d_i|}{n}$$

$$\text{where, } d_i = x_i - \bar{x}$$

ii. **For Grouped data**

If X_1, X_2, X_3,...., X_n are n observations whose corresponding frequencies are f^1, f_2, f_3,...., f_n, then mean deviation from average

$$MD = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i}$$

$$= \frac{\sum_{i=1}^{n} f_i |d_i|}{\sum_{i=1}^{n} f_i}$$

Example 5.3.1 Find the average deviation from mean of the following distribution-

Marks	0-10	10-20	20-30	30-40	40-50
No. of student	5	8	15	16	6

Notes

Solution

Class	Mid val-uex	f	$u = \frac{x-A}{h}$	fu	x-M M =27	$f\|x-M\|$
0-10	5	5	-2	-10	-22	110
10-20	15	8	-1	-8	-12	96
20-30	25	15	0	0	-2	30
30-40	35	16	1	16	8	128
40-50	45	6	2	12	18	108

Here, assumed mean A = 25

Arithmetic mean,

$$M = A + \frac{\sum fu}{N} \times h$$

$$= 25 + \frac{10 \times 10}{50}$$

$$= 25 + 2$$

$$= 27$$

Mean deviation from Mean

$$MD = \frac{\sum f|x-M|}{N}$$

$$= \frac{472}{50}$$

$$= 9.44$$

4. Standard deviation

The standard deviation of a variate is the square root of the arithmetic mean of the squares of all deviations of the values of the variate x from the arithmetic mean of the observations and is denoted by σ.

i. Standard Deviation for Individual Series

If $X_1, X_2, X_3, \ldots, X_n$ are n observations, then

$$SD(\sigma) = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$$

or

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n} - (\bar{x})^2}$$

or

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2}$$

where, $d_i = x_i - A$ and A is assumed mean

ii. Standard Deviation for Frequency Distribution

If $X_1, X_2, X_3, \ldots, X_n$ are n observations whose corresponding frequencies are $f^1, f_2, f_3, \ldots, f_n$, then standard deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{N}}$$

Or

$$\sigma = \sqrt{\frac{1}{N}\left(\sum_{i=1}^{n} f_i d_i^2\right) - \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)^2}$$

where, $d_i = x_i - A$ and $N = \sum f_i$

iii. Standard Deviation for Continous Series

$$\sigma = \sqrt{\frac{1}{N}\sum fu^2 - \left(\frac{\sum fu}{N}\right)^2} \times h$$

Where h is class interval

5.3.3 Variance

The arithmetic mean of the squares of the deviations from arithmetic mean, M of series is called variance and is denoted by σ^2.

Notes

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - M)^2}{n}, \text{ for individual series}$$

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i(x_i - M)^2}{N}, \text{ for grouped series}$$

5.3.4 Properties of Standard Deviation

1. It is independent on origin.
2. It is dependent of change of scale.
3. It is not less than mean deviation from mean.
4. Let ~~n1~~ ~~and~~ n2 be the sizes of the two series. Their means and standard deviations are x_1, x_2 and σ_1, σ_2 respectively. Let x denote the combined mean of two series, that is

$$\bar{x} = \frac{n_1\overline{x_1} + n_2\overline{x_2}}{n_1 + n_2}$$

Hence, the combined standard deviation of two series is given by

$$\sigma = \sqrt{\frac{n_1\left(\sigma_1^{\,2} + d_1^{\,2}\right) + n_2\left(\sigma_2^{\,2} + d_2^{\,2}\right)}{n_1 + n_2}}$$

where,

$$d_1 = \overline{x_1} - \bar{x} \text{ and } d_2 = \overline{x_2} - \bar{x}$$

5. $\sigma = \dfrac{3(\text{Quartile deviation})}{2}$

6. $\sigma = \dfrac{5(\text{Mean deviation})}{2}$

5.3.5 The Coefficient of Variation (C.V.)

The coefficient of variation of given data is the ratio of the standard deviation to the arithmetic mean. The coefficient of variation for given sample can be defined as

$$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100$$

The coefficient of variation is particularly useful when we must compare the variabilities of data sets that are measured in different units.

Notes

Example 5.3.2 Calculate the standard deviation and variation from the following data: 14, 22, 9, 15, 20, 17, 12, 11.

Solution:

Value X	$x-\bar{x}$	$\left(x-\bar{x}\right)^2$
14	-1	1
22	7	49
9	-6	36
15	0	0
20	5	25
17	2	4
12	-3	9
11	-4	16
$\sum x = 120$		$\sum\left(x-\bar{x}\right)^2$

$$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{120}{8}$$

$$= 15$$

$$\sigma = \sqrt{\frac{\sum\left(x-\bar{x}\right)^2}{n}}$$

$$= \sqrt{\frac{140}{8}}$$

$$= \sqrt{17.5}$$

$$= 4.18$$

$$\sigma^2 = 17.47$$

Hence the standard deviation is 4.18 and variation is 17.47.

Example 5.3.3 Find the standard deviation and variance for the following frequency distribution-

Marks	0	10	20	30	40	50	60	70
No. of students	97	90	75	51	25	15	5	2

Notes

Solution

Marks x	Number of student f	fx	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
0	97	0	-17	289	28033
10	90	900	-7	49	4410
20	75	1500	3	9	675
30	51	1530	13	169	8619
40	25	1000	23	529	13225
50	15	750	33	1089	16335
60	5	300	43	1849	9245
70	2	140	53	2809	5618
	$\sum f = 360$	$\sum fx$=6120			$\sum f(x-\bar{x})^2 = 86160$

Arithmetic mean,

$$\bar{x} = \frac{\sum fx}{\sum f}$$
$$= \frac{6120}{360}$$
$$= 17$$

Standard deviation is given by

$$\sigma = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}}$$
$$= \sqrt{\frac{86160}{360}}$$
$$= \sqrt{239.3333}$$
$$= 15.47$$

So, the variance is 239.33.

Example 5.3.4 Find the standard deviation and variance for the following data which represents the wages of 230 workers

Wages (in Rs.)	No. of workers
70-80	12
80-90	18
90-100	35

100-110	42
110-120	50
120-130	45
130-140	20
140-150	8

Solution:

Class	Mid valueX	F	$u=\frac{x-105}{10}$	Fu	fu2
70-80	75	12	-3	-36	108
80-90	85	18	-2	-36	72
90-100	95	35	-1	-35	35
100-110	105	42	0	0	0
110-120	115	50	1	50	50
120-130	125	45	2	90	180
130-140	135	20	3	60	180
140-150	145	8	4	32	128
Total		230		125	753

Here assumed mean A is 105 and class interval is 10.

$$\text{Standard deviation}\,\sigma=\sqrt{\frac{1}{N}\sum fu^2-\left(\frac{\sum fu}{N}\right)^2}\times h$$

$$=\sqrt{\frac{753}{230}-\left(\frac{125}{230}\right)^2}\times 10$$

$$=\sqrt{3.27-0.29}\times 10$$

$$=\sqrt{2.98}\times 10$$

$$=1.73\times 10$$

$$=17.3\text{ Rs}$$

$$\text{Variance }\sigma^2=299.29$$

Example 5.3.4 The mean and standard deviation of marks obtained by 50 students of a class in three subjects, Mathematics, Physics, and Chemistry are given in the following table

Subject	Mathematics	Physics	Chemistry
Mean	42	32	40.9
Standard deviation	12	15	20

Which subject shows the highest variability in marks and which shows the lowest?

Solution

Here n = 50

Notes

For Mathematics,

Coefficient of variation, $C.V. = \frac{\sigma}{\bar{x}} \times 100$

$$C.V. = \frac{12}{42} \times 100$$
$$= \frac{2}{7} \times 100$$
$$= \frac{200}{7}$$
$$= 28.57$$

For Physics,

Coefficient of variation, $C.V. = \frac{\sigma}{\bar{x}} \times 100$

$$C.V. = \frac{15}{32} \times 100$$
$$= \frac{1500}{32}$$
$$= 46.87$$

For Chemistry,

Coefficient of variation, $C.V. = \frac{\sigma}{\bar{x}} \times 100$

$$C.V. = \frac{20}{40.9} \times 100$$
$$= \frac{2000}{40.9}$$
$$= 48.89$$

$\therefore$ C.V. of Chemistry > C.V. of Physics > C.V. of Mathematics

$\therefore$ Chemistry shows the highest variability and Mathematics shows the least variability.

Activity:

1. Calculate the mean deviation from arithmetic mean, median and mode for the following data-

Class	Frequency
14-15	4
15-16	6
16-17	10
17-18	18
18-19	9
19-20	3

Find the variance, standard deviation and coefficient of variation for the following data-

x	1	2	3	4	5	6
f	31	37	33	30	35	34

During the first 10 weeks of a session the marks of two students, X and Y, taking the course were-

X	58	59	60	54	65	66	52	75	69	52
Y	56	87	89	78	71	73	84	65	66	46

Which of the two you would consider to be more consistent?

Summary:

- The simplest possible measure of dispersion is the range which is the difference between the greatest and the least values of the variable.
- The difference between the upper and lower quartiles i.e., $Q_3 - Q_1$ is known as the interquartile range. Half of the interquartile range is called the quartile deviation or semi-interquartile range and it is denoted by QD.

$$QD = \frac{1}{2}(Q_3 - Q_1)$$

- Mean deviation of a distribution is the arithmetic mean of the absolute deviation of the terms of the distribution from its statistical mean (A.M., median or mode).
- For ungrouped or individual data

$$MD = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

For Grouped data

$$MD = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i}$$

$$MD = \frac{\sum_{i=1}^{n} f_i |d_i|}{\sum_{i=1}^{n} f_i}$$

- Coefficient of MD = $\dfrac{\text{Mean deviation}}{\text{Corresponding average}}$

- The standard deviation of a variate is the square root of the arithmetic mean of the squares of all deviations of the values of the variate x from the arithmetic mean of

Notes

the observations and is denoted by σ.

- Standard deviation for individual series

$$\mathrm{SD}(\sigma)=\sqrt{\frac{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}{n}}$$

or

$$\sigma=\sqrt{\frac{\sum_{i=1}^{n}x_i^2}{n}-\left(\bar{x}\right)^2}$$

or

$$\sigma=\sqrt{\frac{1}{n}\sum_{i=1}^{n}d_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}d_i\right)^2}$$

where, $d_i = x_i - A$

- Standard deviation for frequency distribution

$$\sigma=\sqrt{\frac{\sum_{i=1}^{n}f_i\left(x_i-\bar{x}\right)^2}{N}}$$

or

$$\sigma=\sqrt{\frac{1}{N}\left(\sum_{i=1}^{n}f_id_i^2\right)-\left(\frac{1}{N}\sum_{i=1}^{n}f_id_i\right)^2}$$

where, $d_i = x_i - A$ and $N=\sum f_i$

- The arithmetic mean of the squares of the deviations from arithmetic mean, M of series is called variance and is denoted by σ^2.

$$\sigma^2=\frac{\sum_{i=1}^{n}\left(x_i-M\right)^2}{n}\text{, for individual series}$$

$$\sigma^2=\frac{\sum_{i=1}^{n}f_i\left(x_i-M\right)^2}{N}\text{, for grouped series}$$

- Properties of standard deviation
  a. It is independent on origin.
  b. It is dependent of change of scale.
  c. It is not less than mean deviation from mean.

d. Let n1 and n2 be the sizes of the two series. Their means and standard deviations are $\overline{x}_1, \overline{x}_2$ and σ_1, σ_2 respectively. Let $\overline{x}_1$ denote the combined mean of two series, that is

$$\overline{x} = \frac{n_1\overline{x}_1 + n_2\overline{x}_2}{n_1 + n_2}$$

Hence, the combined standard deviation of two series is given by

$$\sigma = \sqrt{\frac{n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1 + n_2}}$$

where,

$$d_1 = \overline{x}_1 - \overline{x} \text{ and } d_2 = \overline{x}_2 - \overline{x}$$

e. $\sigma = \dfrac{3(\text{Quartile deviation})}{2}$

f. $\sigma = \dfrac{5(\text{Mean deviation})}{2}$

- The coefficient of variation of the given data is the ratio of the standard deviation to the arithmetic mean. The coefficient of variation for given sample can be defined as

$$\text{C.V.} = \frac{\sigma}{\overline{x}} \times 100$$

Notes

Unit - 5.4: Skewness and Kurtosis

Recall Session:

In the previous unit, you studied about:

- The Measure of Dispersion
- The Variance
- Properties of Standard Deviation
- The Coefficient of Variation

Unit Outcome:

At the end of this unit, you will learn

1. Define Skewness
2. Define Measures of Skewness
3. Define Kurtosis

5.4.1 Introduction

In the previous unit we studied about measure of dispersion, variance, properties of standard deviation and the coefficient of variation. In this unit we will discuss about skewness, measures of skewness and kurtosis.

5.4.2 Skewness

Skewness denotes the opposite of symmetry. It is lack of symmetry. As applied to frequency distribution it indicates that the distribution of items on it is not symmetrical.

In a symmetrical series the mode, the median and the arithmetic mean are identical. Therefore, skewness or lack of symmetry in a series is shown when these three averages do not coincide.

Skewness can be positive as well as negative. If the mean is greater than the mode or the median, the skewness is positive. If it is less skewness is negative. In other words, if Mode < Median < Mean, then skewness is positive and if Mean < Median < Mode, the skewness is negative.

5.4.3 Measures of Skewness

The first coefficient of skewness is defined by Bowley as-

$$\text{Coeff. of Skewness} = \frac{Q_3 + Q_1 - 2\text{Median}}{Q_3 - Q_1}$$

This expression can be shown as

$$\text{Coeff. of Skewness} = \frac{(Q_3 - \text{Median}) - (\text{Median} - Q_1)}{(Q_3 - \text{Median}) + (\text{Median} - Q_1)}$$

Notes

Bowley's coefficient of skewness is also called quartile coefficient of skewness.

The second coefficient of skewness is defined by Karl Pearson as:

$$\text{Coeff. of Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}}$$

$$= \frac{M - M_o}{\sigma}$$

These coefficients are pure numbers since both numerator and denominator have the same dimensions. The value of the first coefficient lies between -1 and +1 and that of the second lies between -3 and +3.

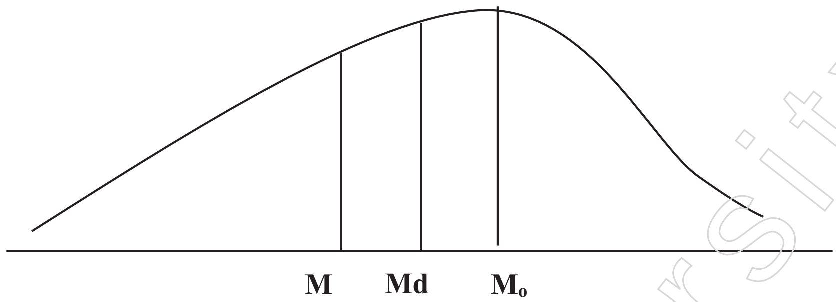


*Fig.*5.4.1 Negative skewness

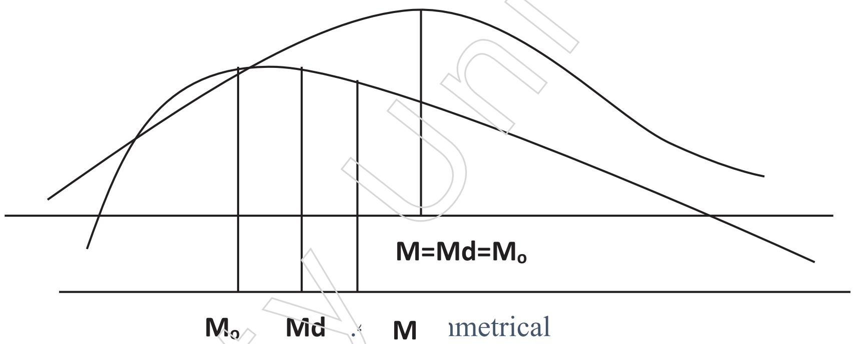


*Fig.*5.4.3 Positive skewness

5.4.4 Kurtosis

The characteristics related with the nature of the concentration of the items in the central part of a frequency distribution is called a Kurtosis.

In other words, Kurtosis is the degree of peakedness (or flatness) in a curve of the frequency distribution. In fact Kurtosis is an indication for the peakedness of a single humped frequency curve β_2, γ_2 measures of Kurtosis indicate the degree to which a curve of the frequency distribution is peaked or flat topped.

Karl Pearson in 1905 introduced the three terms

a. Mesokurtic

Notes

b. Leptokurtic

c. Platykurtic

a. A frequency curve which is not very peaked or very flat topped is called Mesokurtic or normal curve. For such types of curves, $\beta_2 = 3$ and $\gamma_2 = 0$.

b. A frequency curve which is more peaked than the mesokurtic is called Leptokurtic. For such types of curves, $\beta_2 > 3$ and $\gamma_2 > 0$.

c. A frequency curve for which flatness of the top is more than the mesokurtic is called Platykurtic. For such types of curves, $\beta_2 < 3$ and $\gamma_2 < 0$.

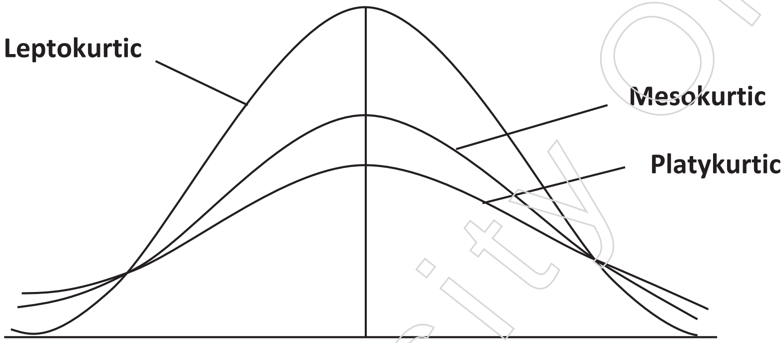


Fig.5.4.4 Comparative picture of three types of Kurtosis

Summary:

- Skewness denotes the opposite of symmetry. It is lack of symmetry.
- Skewness can be positive as well as negative.
- If Mode < Median < Mean, then skewness is positive and if Mean < Median < Mode, the skewness is negative.
- The first coefficient of skewness is defined by Bowley as-

$$\text{Coeff. of Skewness} = \frac{Q_3 + Q_1 - 2\text{Median}}{Q_3 - Q_1}$$

or

$$\text{Coeff. of Skewness} = \frac{(Q_3 - \text{Median}) - (\text{Median} - Q_1)}{(Q_3 - \text{Median}) + (\text{Median} - Q_1)}$$

- The second coefficient of skewness is defined by Karl Pearson as:

$$\text{Coeff. of Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}}$$

$$= \frac{M - M_o}{\sigma}$$

- The value of the first coefficient lies between -1 and +1 and that of the second lies between -3 and +3.
- Kurtosis is the degree of peakedness (or flatness) in a curve of the frequency distribution.
- A frequency curve which is not very peaked or very flat topped is called Mesokurtic or normal curve. For such types of curves, $\beta_2 = 3$ and $\gamma_2 = 0$.

- A frequency curve which is more peaked than the mesokurtic is called Leptokurtic. For such types of curves, $\beta_2 > 3$ and $\gamma_2 > 0$.
- A frequency curve for which flatness of the top is more than the mesokurtic is called Platykurtic. For such types of curves, $\beta_2 < 3$ and $\gamma_2 < 0$.

Further Reading:

1. M. Ray & Har Swarup Sharma, "Mathematical Statistics", Ram Prasad Publication, Agra-3

Exercise

Check your progress:

1. The arithmetic mean of an AP and the mean of first and last term of an AP
   (a) equal
   (b) unequal
   (c) square of each other
   (d) none of the above
2. The mean height of 25 male workers in a factory is 61 cm and the mean height of 35 female workers in the same factory is 58 cm. the combined mean height of 60 workers in the factory is
   (a) 59.25
   (b) 59.5
   (c) 59.75
   (d) 58.75
3. The average of the square of the numbers 0, 1, 2, 3, 4,……., n is
   (a) $\frac{1}{2}n(n+1)$
   (b) $\frac{1}{6}n(2n+1)$
   (c) $\frac{1}{6}(n+1)(2n+1)$
   (d) $\frac{1}{6}n(n+1)$
4. Mean of 100 items is 49. It was discovered that three items which should have been 60, 70 and 80 were wrongly read as 40, 20 and 50, respectively. The correct mean is
   (a) 48
   (b) 89
   (c) 50

Notes

(d) 80

5. The AM of n numbers of a series is . If the sum of first (n -1) terms is k, then the nth number is

   (a) $\bar{x}-k$

   (b) $n\bar{x}-k$

   (c) $\bar{x}-nk$

   (d) $n\bar{x}-nk$

6. If the average of the numbers 148, 146, 144, 142, ….. in AP, be 125, then the total numbers in the series will be

   (a) 18

   (b) 24

   (c) 30

   (d) 48

7. The mean of the values of 1, 2, 3, ……, n with respectively frequencies x, 2x, 3x, ….., nx is

   (a) $\frac{n}{2}$

   (b) $\frac{1}{3}(2n+1)$

   (c) $\frac{1}{6}(2n+1)$

   (d) $\frac{3n}{2}$

8. The mean of n items is . If the first term is increased by 1, second by 2 and so on, then the new mean is

   (a) $\bar{x}+n$

   (b) $\bar{x}+\frac{n}{2}$

   (c) $\bar{x}+\frac{n+1}{2}$

   (d) None of these

Notes

9. If the mean of n observations $1^2, 2^2, 3^2, \ldots, n^2$ is $\frac{46n}{11}$, then n is equal to
   (a) 11
   (b) 12
   (c) 23
   (d) 22

10. The AM of the series 1, 2, 4, 8, 16, ……, 2^n is
   (a) $\frac{2^n - 1}{n}$
   (b) $\frac{2^{n+1} - 1}{n+1}$
   (c) $\frac{2^n + 1}{n}$
   (d) $\frac{2^n - 1}{n+1}$

11. In a class of 50 students, 10 have failed and their average marks are 28. The total marks obtained by the entire class are 2800. The average marks of those who have passed, are
   (a) 43
   (b) 53
   (c) 63
   (d) 70

12. The mean of 30 given numbers, when it is given that the mean of 10 of them is 12 and the mean of the remaining 20 is 9, is equal to
   (a) 11
   (b) 10
   (c) 9
   (d) 5

Notes

13. The mean of a set of observations is . If each observation is divided by, and then is increased by 10, then the mean of the new set is

    (a) $\frac{\bar{x}}{\alpha}$

    (b) $\frac{\bar{x}+10}{\alpha}$

    (c) $\frac{\bar{x}+10\alpha}{\alpha}$

    (d) $\alpha\bar{x}+10$

14 If the sum of deviations of a number of observations about 3 is 40. Then, mean of the observation is

    (a) 7

    (b) 10

    (c) 11

    (d) None of these

15 Geometric mean of first group of 5 observations is 8. And that of second group of 4 observations is $128\sqrt{2}$.

    (a) 64

    (b) $32\sqrt{2}$

    (c) 32

    (d) None of these

16 Which of the following is not a measure of central tendency?

    (a) Mean

    (b) Median

    (c) Mean-deviation

    (d) Mode

17 For dealing with qualitative data the best average is

    (a) AM

    (b) GM

    (c) Median

    (d) Mode

18 Coefficient of skewness for the values

    Median = 18.8, Q1 = 14.6, Q3 = 252 is

    (a) 0.2

(b) 0.5

(c) 0.7

(d) None of these

19. If the sum of 11 consecutive natural numbers is 2761, then the middle number is

(a) 249

(b) 250

(c) 251

(d) 252

20. If the mode of the data is 18 and the mean is 24, then median is

(a) 18

(b) 24

(c) 22

(d) 21

21. If in a moderately asymmetrical distribution mode and mean of the data are 6M and 9M respectively, then median is

(a) 8M

(b) 7M

(c) 6M

(d) 5M

22. If the median of 21 observations is 40 and if the observations greater than the median are increased by 5, then the median of the new data will be

(a) 40

(b) 45

(c) $4+\frac{50}{21}$

(d) $45-\frac{50}{21}$

23. In a moderately skewed distribution, the values of mean and median are 5 and 6, respectively. The value of mode in such a situation is approximately equal to

(a) 8

(b) 11

(c) 16

(d) None of these

24. Consider the following statements

i. The values of median and mode can be determined graphically.

ii. Mean, median and mode have the same unit.

Notes

iii. Range is the best measure of dispersion.

iv. Which of these is/are correct?

(a) Only I

(b) Only II

(c) Both II and III

(d) None of these

25. The quartile deviation of daily wages (in rupees) of 7 persons given below 12, 7, 15, 10, 17, 19 and 25 is

(a) 14.5

(b) 5

(c) 9

(d) 4.5

26. If two variables x and y are such that and quartile deviation(QD) of x is 8, then QD of y is

(a) 2

(b) 8

(c) 4

(d) None of these

27. For a series the value of mean deviation is 15, the most likely value of its quartile deviation is

(a) 12.5

(b) 11.6

(c) 13

(d) 9.7

28. When tested, the lives (in hours) of 5 bulbs were noted as follows

1357, 1090, 1666, 1494, 1623

The mean deviations (in hours) from their mean is

(a) 178

(b) 179

(c) 220

(d) 356

29. The SD of 15 items is 6 and if each item is 6 and if each item is decreases by 1, then standard deviation will be

(a) 5

(b) 7

(c) 8

(d) 6

30. If SD of X is S, then SD of the variable $\mu = \frac{aX+b}{c}$, where a, b and c are constants, is

(a) $\left|\frac{c}{a}\right|\sigma$

(b) $\left|\frac{a}{c}\right|\sigma$

(c) $\left|\frac{b}{c}\right|\sigma$

(d) $\frac{c^2}{a^2}\sigma$

31. If $\overline{x}$ is the arithmetic mean of n independent variates x_1, x_2, x_3,....x_n each of the standard deviation , then variance is

(a) $\frac{\sigma^2}{n}$

(b) $\frac{n\sigma^2}{2}$

(c) $\frac{(n+1)\sigma^2}{3}$

(d) None of these

32. If the variance of 1,2,3,4,.....,10 is $\frac{99}{12}$,, then the standard deviation of 3, 6, 9, 12, 30 is

(a) $\frac{297}{4}$

(b) $\frac{3}{2}\sqrt{33}$

(c) $\frac{3}{2}\sqrt{99}$

(d) $\sqrt{\frac{99}{12}}$

33. The mean and variance of n values of a variable x are 0 and , respectively. If the variable , then mean of y is

(a) σ

Notes

Notes

(b) σ^2

(c) 1

(d) None of these

34. Two samples of sizes 100 and 150 have means 45 and 55 and standard deviation 7 and 12, respectively. Find the mean and standard deviation of the combined sample

    (a) 11, 30

    (b) 11, 45

    (c) 11, 40

    (d) 11, 50

35. If the mean of five observations x, x + 2, x + 4, x + 6 and X + 8 is 11, then the mean of last three observations is

    (a) 13

    (b) 15

    (c) 17

    (d) None of these

36. If for a distribution the difference of first quartile and median is greater than difference of median and third quartile then distribution is classified as

    (a) absolute open ended

    (b) positively skewed

    (c) negatively skewed

    (d) not skewed at all

37. if the first quartile and third quartile are as 32 and 35 respectively with the median of 20 then distribution is skewed to

    (a) lower tail

    (b) upper tail

    (c) close end tail

    (d) open end tail

38. the measurement techniques used to measure the extent of skewness in data set values are called

    (a) measure of distribution width

    (b) measure of median tail

    (c) measure of tail distribution

    (d) measure of skewness

39. The statistical measure such as average deviation, standard deviation and mean are classified as part of

    (a) deciles system

Notes

(b) moment system

(c) percentile system

(d) quartile system

40. The method of calculating skewness which is based on the positions of quartiles and median in a distribution is called

(a) Gary's coefficient of skewness

(b) Sharma's coefficient of skewness

(c) Bowley's coefficient of skewness

(d) Jack Karl's coefficient of skewness

41. The median of a moderately skewed distribution is 8, third quartile is 12, first quartile is 8 and inter-quartile range is 4 then relative coefficient of skewness is

(a) ± 8

(b) ± 1

(c) ± 9

(d) ± 11

42. For the Karl Pearson's skewness coefficient, the value of skewness must be in limits

(a) ± 3

(b) ± 5

(c) ± 4

(d) ± 1

43. The distribution is considered leptokurtic if

(a) β_3 is less than three

(b) β_2 is greater than two

(c) β_3 is greater than three

(d) β_3 is greater than three

44. The distribution is considered platykurtic if

(a) β_3 is less than three

(b) β_2 is greater than two

(c) β_3 is greater than three

(d) β_3 is greater than three

45. In kurtosis, the frequency curve that has flatten top normal curve of bell shaped distribution is classified as

(a) Leptokurtic

(b) Platykurtic

(c) Mega curve

Notes

(d) Mesokurtic

Answer Keys (Exercise):

Question	Answer	Question	Answer	Question	Answer
1	a	2	a	3	b
4	c	5	b	6	b
7	b	8	c	9	a
10	b	11	c	12	b
13	c	14	a	15	c
16	c	17	c	18	a
19	c	20	c	21	a
22	a	23	a	24	a
25	d	26	d	27	a
28	a	29	d	30	b
31	a	32	b	33	b
34	c	35	a	36	b
37	a	38	d	39	b
40	c	41	b	42	a
43	d	44	c	45	b